Decidable Fragments of First-order Modal Logic

Anantha Padmanabha
IIT Madras

joint work with Varad Joshi, Mo Liu, R. Ramanujam and Yanjing Wang

CAALM
LIPN, Université Sorbonne Paris Nord

5 June 2025

Propositional Modal Logic

- Extends Propositional Logic with Modal Operators $\Box$ and $\Diamond$
- Generalizes various logics

Logic Variant	Modal Interpretation
Temporal Logic	Time
Epistemic Logic	Knowledge
Doxastic Logic	Belief
Deontic Logic	Obligations
Dynamic Logic	Actions

- Example : $\Box(p) \vee \Diamond(q)$

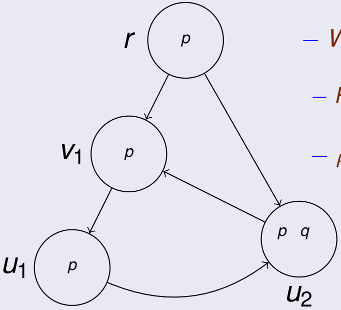
Propositional Modal Logic

Syntax

$$\varphi \; := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \Diamond\varphi$$

Structures

A Kripke structure is given by $M = (W, R, \rho)$



- W is non-empty set of worlds
- $R \subseteq (W \times W)$ is the accessiblity relation
- $\rho : W \mapsto 2^{\mathcal{P}}$ is the valuation function

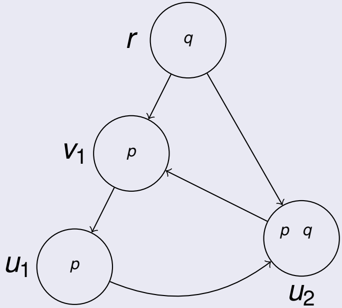
Propositional Modal Logic

Syntax

$$\varphi \; := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \Diamond\varphi$$

Semantics

A Kripke structure is given by $M = (W, R, \rho)$



$M, w \models p$ if $p \in \rho(w)$

$M, w \models \Box\varphi$ if for every $w' \in W$ if $(w, w') \in R$ then $M, w' \models \varphi$

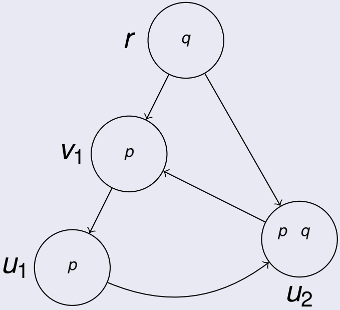
Propositional Modal Logic

Syntax

$$\varphi \ := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \Diamond\varphi$$

Semantics

A Kripke structure is given by $M = (W, R, \rho)$



$M, w \models p$ if $p \in \rho(w)$

$M, w \models \Box\varphi$ if for every $w' \in W$ if $(w, w') \in R$ then $M, w' \models \varphi$

$M, w \models \Diamond\varphi$ if there is some $(w, w') \in R$ st $M, w' \models \varphi$

Propositional Modal Logic

Satisfiability Problem

- Given a formula φ, is there some model M and w such that $M, w \models \varphi$?
- Also called the Synthesis Problem

Propositional Modal Logic

Satisfiability Problem

- Given a formula φ, is there some model M and w such that $M, w \models \varphi$?
- Also called the Synthesis Problem

- Decidable (PSPACE-complete)

Propositional Modal Logic

Satisfiability Problem

- Given a formula φ, is there some model M and w such that $M, w \models \varphi$?
- Also called the Synthesis Problem

- Decidable (PSPACE-complete)
- Can be embedded into
  - [Scott; Kolaitis et.al] Two variable fragment of FO
  - [Andreka et.al,] Guarded fragment of FO
  - [Pratt-Hartmann et. al] Fluted fragment of FO

First Order Modal Logic

- Extends First Order Logic with Modal Operators.

First Order Modal Logic

- Extends First Order Logic with Modal Operators.
- Example: $\forall x \Box (P(x)) \vee \exists y (\Diamond Q(x, y))$

Variant	Interpretation
Time	For every process, the process will always access only its local variables or it synchronizes with another process in some future
Epistemic Logic	For every x, Alice knows that x is at the party or considers it possible that x has quarrelled with some y

- Also suitable to model evolving graphs, databases etc.

First Order Modal Logic

Syntax

$$\varphi \ := \ P(\overline{x}) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x \ \varphi \mid \Box\varphi \mid \Diamond\varphi$$

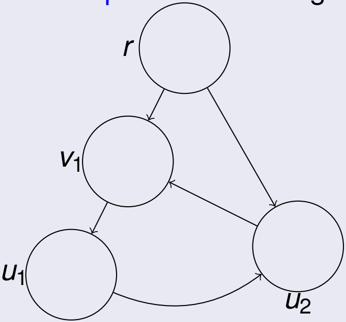
First Order Modal Logic

Syntax

$$\varphi \;:=\; P(\overline{x}) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x\ \varphi \mid \Box\varphi \mid \Diamond\varphi$$

Semantics

A Kripke structure is given by $M = (W, D, \delta, R, \rho)$



– W is non-empty set of worlds

– $R \subseteq (W \times W)$ is the accessiblity relation

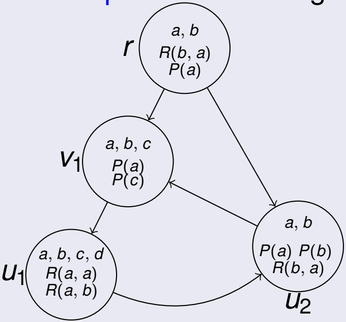
First Order Modal Logic

Syntax

$$\varphi \;:=\; P(\overline{x}) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x\ \varphi \mid \Box\varphi \mid \Diamond\varphi$$

Semantics

A Kripke structure is given by $M = (W, D, \delta, R, \rho)$

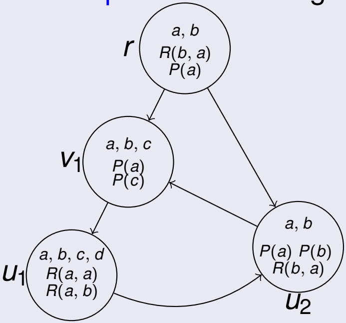


- W is non-empty set of worlds
- $R \subseteq (W \times W)$ is the accessiblity relation
- D is non-empty set of Domain
- $\delta : W \mapsto 2^D$ is the Local Domain
- ρ gives valuation for predicates at every world over local domain

First Order Modal Logic

Structure

A Kripke structure is given by $M = (W, D, \delta, R, \rho)$



$M, w, \sigma \models P(\overline{x})$ if $\sigma(\overline{x}) \in \rho(w, P)$

$M, w, \sigma \models \Box\varphi$ if for every $w' \in W$ if $(w, w') \in R$ then $M, w', \sigma \models \varphi$

$M, w, \sigma \models \exists x\ \varphi$ if there exists $a \in \delta(w)$ st $M, w, \sigma_{[x \to a]} \models \varphi$

First Order Modal Logic

Conditions on Local Domain

- Increasing Domain models: If $(u, v) \in R$ then $\delta(u) \subseteq \delta(v)$

First Order Modal Logic

Conditions on Local Domain

- Increasing Domain models: If $(u, v) \in R$ then $\delta(u) \subseteq \delta(v)$
- Constant Domain models: Local Domain is same at all worlds

Satisfiability Problem

Given a FOML formula φ, is there some model M, w and σ such that $M, w, \sigma \models \varphi$?

Satisfiability Problem

Given a FOML formula φ, is there some model M, w and σ such that $M, w, \sigma \models \varphi$?

Bad news

- First Order Logic is a fragment of First Order Modal Logic

Satisfiability Problem

Given a FOML formula φ, is there some model M, w and σ such that $M, w, \sigma \models \varphi$?

Bad news

- First Order Logic is a fragment of First Order Modal Logic
- What about Modal extensions of decidable fragments of FO?
  Like Unary predicates, 2 variable fragment, Guarded, Fluted...

Satisfiability problem for FOML

- Undecidable even when atoms are restricted to unary predicates [$P(x), Q(x)..$].
  Note: FO with unary predicates is decidable.

Satisfiability problem for FOML

- Undecidable even when atoms are restricted to unary predicates [$P(x), Q(x)..$].
  Note: FO with unary predicates is decidable.

Proof [Kripke, 1962]

- Reducing satisfiability of FO(R) [where R is a binary predicate] to FOML satisfiability.

Satisfiability problem for FOML

- Undecidable even when atoms are restricted to unary predicates [$P(x), Q(x)..$].
  Note: FO with unary predicates is decidable.

Proof [Kripke, 1962]

- Reducing satisfiability of FO(R) [where R is a binary predicate] to FOML satisfiability.
- Translate $R(x, y)$ as $\Diamond[P(x) \wedge Q(y)]$.

  Ex. $\forall x \exists y \forall z \ (R(x, y) \rightarrow \neg R(y, z))$ is translated to
  $\forall x \exists y \forall z \ (\Diamond(P(x) \wedge Q(y)) \rightarrow \neg\Diamond(P(y) \wedge Q(z)))$.

Satisfiability problem for FOML

- Undecidable even when atoms are restricted to unary predicates $[P(x), Q(x)..]$.
  Note: FO with unary predicates is decidable.

Proof [Kripke, 1962]

- Reducing satisfiability of $FO(R)$ [where R is a binary predicate] to FOML satisfiability.
- Translate $R(x, y)$ as $\Diamond[P(x) \wedge Q(y)]$.

  Ex. $\forall x \exists y \forall z \; (R(x, y) \rightarrow \neg R(y, z))$ is translated to
  $\forall x \exists y \forall z \; (\Diamond(P(x) \wedge Q(y)) \rightarrow \neg \Diamond(P(y) \wedge Q(z)))$.
- Any $FO(R)$ formula is satisfiable iff its translated FOML formula is satisfiable.

Satisfiability Problem

More bad news!

Satisfiability Problem

More bad news!

All these fragments of FOML are undecidable:

Unary Predicates	Kripke	1962
Two variable fragment of FOML	Wolter & Zakharyaschev	2001
Guarded fragment of FOML		

Satisfiability Problem

More bad news!

All these fragments of FOML are undecidable:

Unary Predicates	Kripke	1962
Two variable fragment of FOML	Wolter & Zakharyaschev	2001
Guarded fragment of FOML		

Are there reasonable decidable fragments at all?

Source of Undecidability

- Two or more free variables inside the scope of modal operators.
- Unrestricted occurrence of quantifiers and modalities.

Can we restrict the syntax to avoid these to get decidable fragments?

Source of Undecidability

- Two or more free variables inside the scope of modal operators.
- Unrestricted occurrence of quantifiers and modalities.

Can we restrict the syntax to avoid these to get decidable fragments?

Monodic fragment

Monodic fragment

- Every modal formula has at most 1 free variable.

Monodic fragment

- Every modal formula has at most 1 free variable.
- Eg. $\forall x.\exists y \left(\Box P(x) \rightarrow \Diamond(Q(y) \vee Q(x))\right)$ is NOT monodic formula

Monodic fragment

- Every modal formula has at most 1 free variable.
- Eg. $\forall x.\exists y \left(\Box P(x) \rightarrow \Diamond(Q(y) \vee Q(x))\right)$ is NOT monodic formula
- Eg. $\forall x. \left(\Box P(x) \rightarrow \Diamond(\exists y.\ R(x, y))\right)$ is monodic formula

Monodic fragment

- Every modal formula has at most 1 free variable.
- Eg. $\forall x.\exists y \left(\Box P(x) \rightarrow \Diamond(Q(y) \vee Q(x))\right)$ is NOT monodic formula
- Eg. $\forall x. \left(\Box P(x) \rightarrow \Diamond(\exists y.\ R(x, y))\right)$ is monodic formula

Theorem (Wolter, Zakharyashev; 2001)

Monodic FOML over unary predicates (unary predicates, two variables...) is decidable.

Source of Undecidability

- Two or more free variables inside the scope of modal operators.
- Unrestricted occurrence of quantifiers and modalities.

Can we restrict the syntax to avoid these to get decidable fragments?

Bundled fragment of FOML

Syntax

Given $\mathcal{V}$ (variables) and $\mathcal{P}$ (predicates), the bundled fragment of FOML denoted by Bundle is defined as follows:

$$\varphi ::= P\overline{x} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x \Box \varphi \mid \forall x \Box \varphi$$

where $x \in \mathcal{V}$, $P \in \mathcal{P}$.

Bundled fragment of FOML

Syntax

Given $\mathcal{V}$ (variables) and $\mathcal{P}$ (predicates), the bundled fragment of FOML denoted by Bundle is defined as follows:

$$\varphi ::= P\overline{x} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x \Box \varphi \mid \forall x \Box \varphi$$

where $x \in \mathcal{V}$, $P \in \mathcal{P}$.

- Note that dual of $\exists x \Box$ is given by $\forall x \Diamond \varphi := \neg \exists x \Box \neg \varphi$.
- Similarly dual of $\forall x \Box$ is given by $\exists x \Diamond \varphi := \neg \forall x \Box \neg \varphi$.

Bundled fragment of FOML

Syntax

Given $\mathcal{V}$ (variables) and $\mathcal{P}$ (predicates), the bundled fragment of FOML denoted by Bundle is defined as follows:

$$\varphi ::= P\overline{x} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x \Box \varphi \mid \forall x \Box \varphi$$

where $x \in \mathcal{V}$, $P \in \mathcal{P}$.

- Note that dual of $\exists x \Box$ is given by $\forall x \Diamond \varphi := \neg \exists x \Box \neg \varphi$.
- Similarly dual of $\forall x \Box$ is given by $\exists x \Diamond \varphi := \neg \forall x \Box \neg \varphi$.
- Standard $\Box$ can be defined: $\exists z \Box \varphi$ where $z \notin \mathrm{Fv}(\varphi)$.

Bundled fragment of FOML

- Are bundled fragments interesting at all? What properties can they express?

Bundled fragment of FOML

- Are bundled fragments interesting at all? What properties can they express?
- Bundled fragments are motivated by [Wang] where formulas are restricted to the form $\exists x \Box \varphi$.
  Can model Know how, Know why...

Bundled fragment of FOML

- Are bundled fragments interesting at all? What properties can they express?
- Bundled fragments are motivated by [Wang] where formulas are restricted to the form $\exists x \Box \varphi$.
  Can model Know how, Know why...

Examples

- (Epistemic logic) Agent knows who killed Mary $\exists x \Box (killed(x, Mary))$.
- (Epistemic logic) Agent knows that someone killed Mary $\Box (\exists x\ killed(x, Mary))$.

Bundled fragment of FOML

- Are bundled fragments interesting at all? What properties can they express?
- Bundled fragments are motivated by [Wang] where formulas are restricted to the form $\exists x \Box \varphi$.
  Can model Know how, Know why...

Examples

- (Epistemic logic) Agent knows who killed Mary
  $\exists x \Box (killed(x, Mary))$.
- (Epistemic logic) Agent knows that someone killed Mary
  $\Box (\exists x \; killed(x, Mary))$.
- (Temporal logic) All clients have equal priority.
  $\forall y \Diamond \left(answered(y) \right)$

Decidability results

P., Ramanujam, Wang [2018]

- $\exists x \Box$, $\forall x \Box$ over increasing domain is decidable with arbitrary arity predicates.

Decidability results

P., Ramanujam, Wang [2018]

- $\exists x\Box$, $\forall x\Box$ over increasing domain is decidable with arbitrary arity predicates.
- $\forall x\Box$ over constant domain is undecidable with unary predicates.

Decidability results

P., Ramanujam, Wang [2018]

- $\exists x\Box$, $\forall x\Box$ over increasing domain is decidable with arbitrary arity predicates.
- $\forall x\Box$ over constant domain is undecidable with unary predicates.
- $\exists x\Box$ over constant domain is decidable with arbitrary arity predicates.

Key Idea for (un)decidability

- Consider the formula $\forall x \Diamond \exists y \Diamond R(x, y)$

Key Idea for (un)decidability

- Consider the formula $\forall x \Diamond \exists y \Diamond R(x, y)$

Const. Domain Model

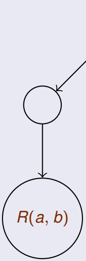
$\mathcal{D} = \{a \qquad\qquad\qquad \}$

Key Idea for (un)decidability

- Consider the formula $\forall x \Diamond \exists y \Diamond R(x, y)$

Const. Domain Model

$\mathcal{D} = \{a \, , b \qquad \}$

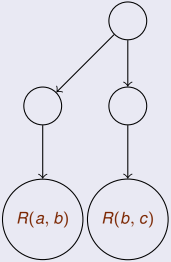


Key Idea for (un)decidability

- Consider the formula $\forall x \Diamond \exists y \Diamond R(x, y)$

Const. Domain Model

$\mathcal{D} = \{a, b, c \quad \}$

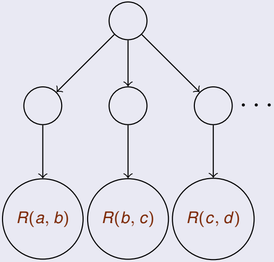


Key Idea for (un)decidability

- Consider the formula $\forall x \Diamond \exists y \Diamond R(x, y)$

Const. Domain Model

$\mathcal{D} = \{a\ , b, c, d, \cdots\}$

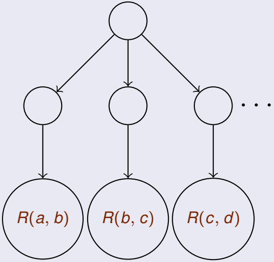


Key Idea for (un)decidability

- Consider the formula $\forall x \Diamond \exists y \Diamond R(x,y)$

Const. Domain Model

$\mathcal{D} = \{a\ , b, c, d, \cdots\}$



Inc. Domain Model

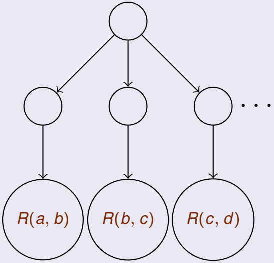
$\mathcal{D} = \{a \quad \}$

Key Idea for (un)decidability

- Consider the formula $\forall x \Diamond \exists y \Diamond R(x, y)$

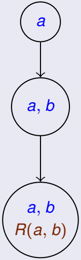
Const. Domain Model

$\mathcal{D} = \{a, b, c, d, \cdots\}$



Inc. Domain Model

$\mathcal{D} = \{a, b\}$



More bundles

More bundles

Syntax

$$\varphi \;:=\; P(\overline{x}) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x \Box \varphi \mid \forall x \Box\, \varphi \mid \Box \exists x \varphi \mid \Box \forall x \varphi$$

More bundles

Syntax

$$\varphi \ := \ P(\overline{x}) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x \Box \varphi \mid \forall x \Box \ \varphi \mid \Box \exists x \varphi \mid \Box \forall x \varphi$$

- Other bundles: $\Box\forall$, $\Box\exists$ and the combinations thereof.

More bundles

Syntax

$$\varphi \ := \ P(\overline{x}) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x \Box \varphi \mid \forall x \Box \ \varphi \mid \Box \exists x \varphi \mid \Box \forall x \varphi$$

- Other bundles: $\Box\forall$, $\Box\exists$ and the combinations thereof.
- Eg. $\exists\Box + \Box\exists$ can express $\Diamond\forall x \exists y \ \Box P(x, y)$

More bundles

Syntax

$$\varphi \ := \ P(\overline{x}) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x \Box \varphi \mid \forall x \Box \ \varphi \mid \Box \exists x \varphi \mid \Box \forall x \varphi$$

- Other bundles: $\Box\forall$, $\Box\exists$ and the combinations thereof.
- Eg. $\exists\Box + \Box\exists$ can express $\Diamond\forall x \exists y \ \Box P(x, y)$
- Can we map the decidability of the combinations of the various bundled fragments?

Decidability of combinations

Liu, P., Ramanujam, Wang, 2023

$\forall\Box$	$\exists\Box$	$\Box\forall$	$\Box\exists$	Over Inc. Domain
✓	✓	✗	✗	decidable
✗	✗	✓	✓	decidable
$\star$	✓	✓	$\star$	Undecidable
✗	✓	✗	✓	Open (No FMP)
✓	✓	✗	✓	Undecidable
✓	✗	✓	✓	decidable

Decidability of combinations

Liu, P., Ramanujam, Wang, 2023

$\forall\Box$	$\exists\Box$	$\Box\forall$	$\Box\exists$	Over Inc. Domain
✓	✓	✗	✗	decidable
✗	✗	✓	✓	decidable
$\star$	✓	✓	$\star$	Undecidable
✗	✓	✗	✓	Open (No FMP)
✓	✓	✗	✓	Undecidable
✓	✗	✓	✓	decidable

- If $\forall x \exists y \Box$ and $\forall x \Box \forall y \Box \forall z \Box$ are expressible then Undecidable

Decidability of combinations

Liu, P., Ramanujam, Wang, 2023

$\forall\Box$	$\exists\Box$	$\Box\forall$	$\Box\exists$	Over Inc. Domain
✓	✓	✗	✗	decidable
✗	✗	✓	✓	decidable
$\star$	✓	✓	$\star$	Undecidable
✗	✓	✗	✓	Open (No FMP)
✓	✓	✗	✓	Undecidable
✓	✗	✓	✓	decidable

- If $\forall x \exists y \Box$ and $\forall x \Box \forall y \Box \forall z \Box$ are expressible then Undecidable
- If $\forall x \exists y \Box$ is not expressible then decidable

Decidability of combinations

Liu, P., Ramanujam, Wang, 2023

$\forall\Box$	$\exists\Box$	$\Box\forall$	$\Box\exists$	Over Inc. Domain
✓	✓	✗	✗	decidable
✗	✗	✓	✓	decidable
$\star$	✓	✓	$\star$	Undecidable
✗	✓	✗	✓	Open (No FMP)
✓	✓	✗	✓	Undecidable
✓	✗	✓	✓	decidable

- If $\forall x \exists y \Box$ and $\forall x \Box \forall y \Box \forall z \Box$ are expressible then Undecidable
- If $\forall x \exists y \Box$ is not expressible then decidable
- If $\forall x \exists y \Box$ is expressible but $\forall x \Box \forall y \Box \forall z \Box$ are expressible then No FMP, but decidability is open

$\exists\Box + \Box\exists$

- Does not satisfy Finite Model Property

$$\varphi := \Diamond\forall x\Big[\ \exists y\Box\Box Pxy \wedge \Box\Box\neg Pxx \wedge \Diamond\forall y\Big(\big[\Diamond Pxy \leftrightarrow \Box Pxy\big] \wedge \Diamond\forall z\big[(Pxy \wedge Pyz) \to (Pxz)\big]\Big)\Big]$$

$\exists\Box + \Box\exists$

- Does not satisfy Finite Model Property

$$\varphi := \Diamond\forall x\Big[\ \exists y\Box\Box Pxy \wedge \Box\Box\neg Pxx \wedge \Diamond\forall y\Big(\big[\Diamond Pxy \leftrightarrow \Box Pxy\big] \wedge \Diamond\forall z\big[(Pxy \wedge Pyz) \to (Pxz)\big]\Big)\Big]$$

Joshi and P., (Arxiv, 2025)

$\exists\Box + \Box\exists$ over increasing domain models is decidable

$\exists\Box + \Box\exists$

- Does not satisfy Finite Model Property

$$\varphi := \Diamond\forall x\Big[\ \exists y\Box\Box Pxy \wedge \Box\Box\neg Pxx \wedge \Diamond\forall y\Big(\big[\Diamond Pxy \leftrightarrow \Box Pxy\big] \wedge \Diamond\forall z\big[(Pxy \wedge Pyz) \rightarrow (Pxz)\big]\Big)\Big]$$

Joshi and P., (Arxiv, 2025)

$\exists\Box + \Box\exists$ over increasing domain models is decidable

- Rare extension of FO that is decidable without Finite Model Property

$\exists\Box + \Box\exists$

- Does not satisfy Finite Model Property

$$\varphi := \Diamond\forall x\Big[\ \exists y\Box\Box Pxy \wedge \Box\Box\neg Pxx \wedge \Diamond\forall y\Big(\big[\Diamond Pxy \leftrightarrow \Box Pxy\big] \wedge \Diamond\forall z\big[(Pxy \wedge Pyz) \rightarrow (Pxz)\big]\Big)\Big]$$

Joshi and P., (Arxiv, 2025)

$\exists\Box + \Box\exists$ over increasing domain models is decidable

- Rare extension of FO that is decidable without Finite Model Property
- Proof introduces a novel way to maintain skolem witnesses in a pseudo-finite way

$\exists\Box + \Box\exists$

- Does not satisfy Finite Model Property

$$\varphi := \Diamond\forall x\Big[\ \exists y\Box\Box Pxy \wedge \Box\Box\neg Pxx \wedge \Diamond\forall y\Big(\big[\Diamond Pxy \leftrightarrow \Box Pxy\big] \wedge \Diamond\forall z\big[(Pxy \wedge Pyz) \rightarrow (Pxz)\big]\Big)\Big]$$

Joshi and P., (Arxiv, 2025)

$\exists\Box + \Box\exists$ over increasing domain models is decidable

- Rare extension of FO that is decidable without Finite Model Property
- Proof introduces a novel way to maintain skolem witnesses in a pseudo-finite way
- Techniques might be useful to prove other logics on tress that violate finite model property

Conclusion

More on Satisfiability problem

- $\Box\exists$ over constant domain models
- Frame restrictions

Conclusion

More on Satisfiability problem

- $\Box\exists$ over constant domain models
- Frame restrictions

Many more questions

- Interpolation
- Definability
- Finite representations / model checking
- Tools

If you stare at any logic long enough, some decidable fragment will stare back at you!