# A Complexity Dichotomy for Semilinear Target Sets in Automata with One Counter

Yousef Shakiba[1] and Henry Sinclair-Banks[2] and Georg Zetzsche[1]

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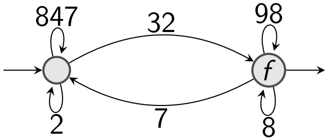
work to appear at LICS 2025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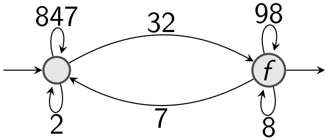


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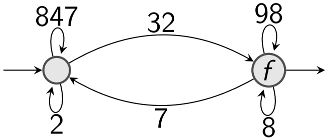
**Given** Automaton, number $t \in \mathbb{N}$

**Question** Can we reach $(f, t)$?

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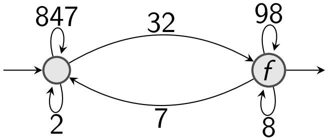


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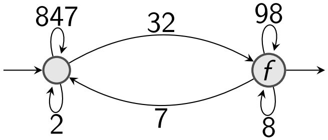


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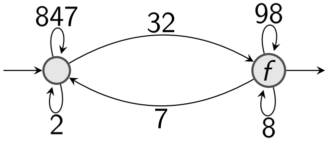


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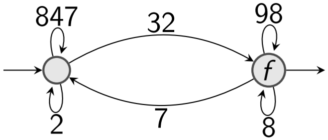


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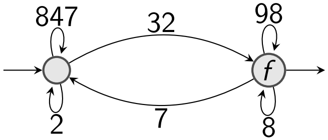


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General toolbox beyond coverability?

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Probability of hitting $S$ within $x + [-n, n]$

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Similar $\mathrm{AC}^1$/NP dichotomies with negative updates:

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- $\mathbb{Z}$-VASS (counters can go negative): modified density notion

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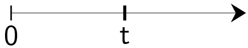

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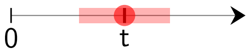

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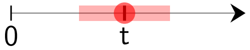


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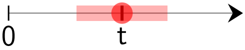


$D(S) = 0$

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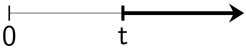

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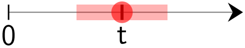


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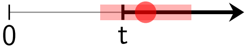

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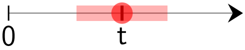


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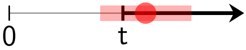


$D(S) = \frac{1}{2}$

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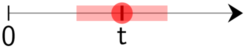


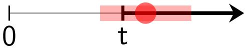


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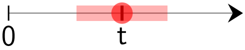


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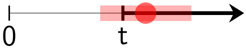


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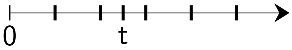
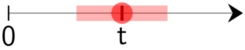


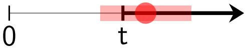

Similarly: $S_{r,s,t} = [t, \infty) \backslash \{r, s\}$

$S_t = \{t\}$




$D(S) = 0$

$S_t = [t, \infty)$




$D(S) = \frac{1}{2}$

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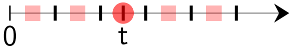


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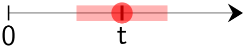


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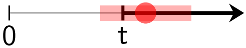


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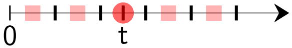


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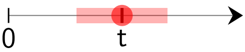


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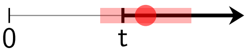


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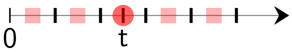


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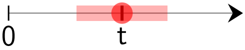


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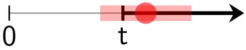


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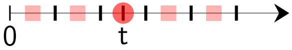


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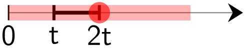

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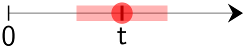


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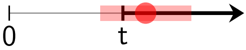


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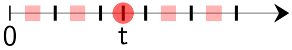


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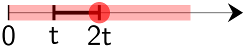


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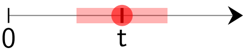


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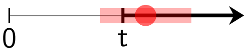


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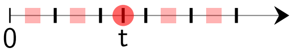


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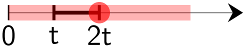


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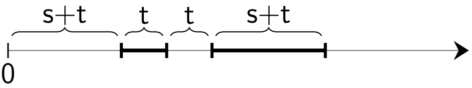

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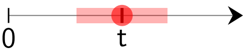


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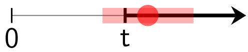


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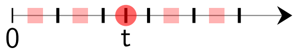


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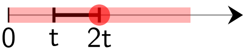


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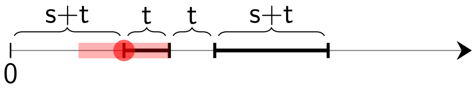

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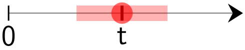


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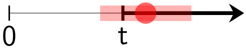


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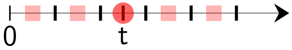


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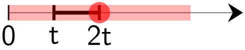


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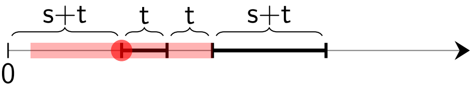

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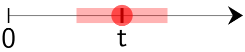


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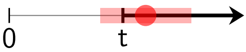


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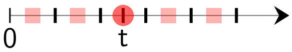


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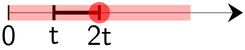


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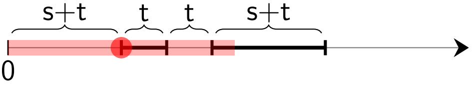

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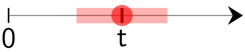


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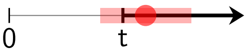


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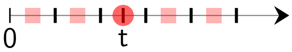


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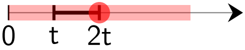


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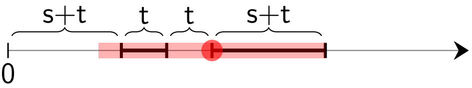

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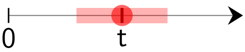


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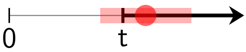


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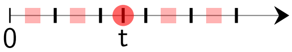


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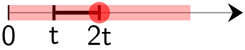


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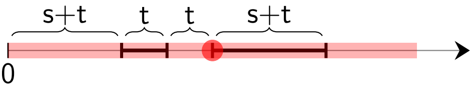

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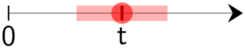


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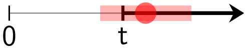


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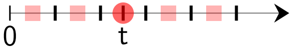


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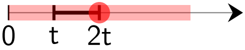


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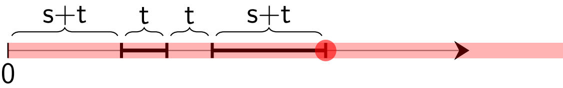

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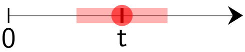


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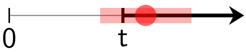


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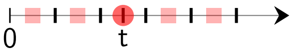


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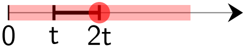


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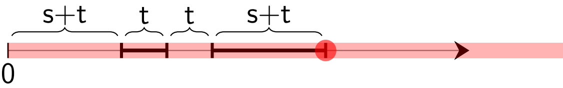


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# How to solve cases with $D(S) > 0$?

Step I: Make automaton acyclic

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Carathéodory bound on integer cones (Eisenbrand & Shmonin 2006)

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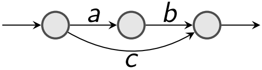


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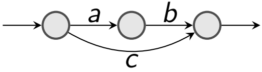


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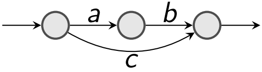


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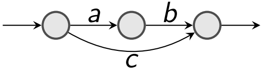


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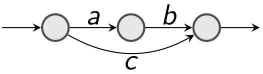


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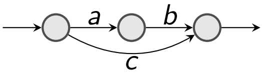


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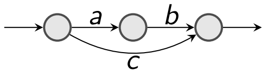


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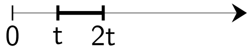
Equivalently: impose $X^i + X^j = X^{\max(i,j)}$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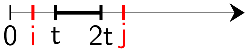


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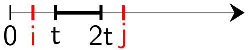


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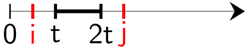


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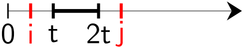


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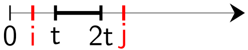


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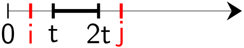


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3 terms that cannot be merged $\rightsquigarrow$ 2 gaps of size $> t$ $\rightsquigarrow$ some term $> 2t$

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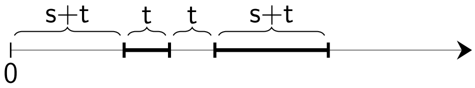

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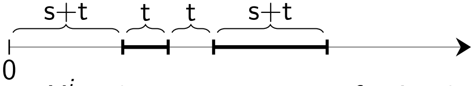


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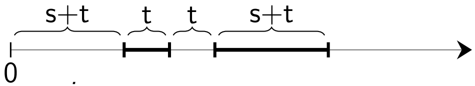


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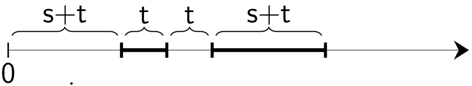


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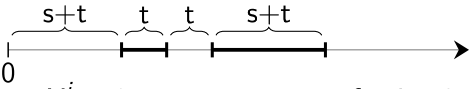


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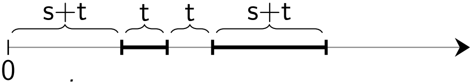


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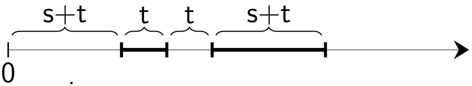


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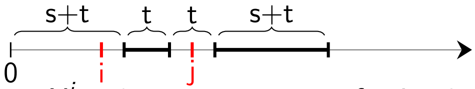


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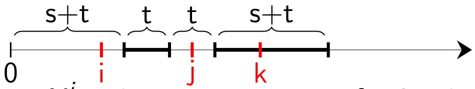


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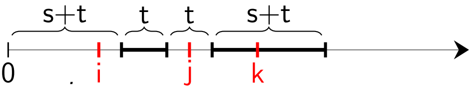


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### Lemma

Three equations $\rightsquigarrow$ bounded number of $X^{[i,j]}$ terms

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*Let $S \subseteq \mathbb{N}^p \times \mathbb{N}$ be Presburger-definable.*

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Toolbox: Weighted automata over suitable semirings

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Whenever $D(S) > 0$, we can find equation system

To get $\mathsf{AC}^1$ upper bound:

- Compute $A^n$ by repeated squaring $\rightsquigarrow$ logarithmic number of steps
- Show that matrix product can be computed in $\mathsf{AC}^0$

### Theorem (Shakiba, Sinclair-Banks, Z. 2025)

*Let $S \subseteq \mathbb{N}^p \times \mathbb{N}$ be Presburger-definable.*

1. *If $D(S) > 0$, then* $\mathrm{Reach}(S)$ *is in* $\mathsf{AC}^1 \subseteq \mathsf{P}$.
2. *Otherwise,* $\mathrm{Reach}(S)$ *is* NP*-complete.*

Similar $\mathsf{AC}^1$/NP dichotomies:

- $\mathbb{Z}$-VASS (counters can go negative): modified density notion
- VASS (must stay non-negative): "uniformly quasi-upward closed"
  In particular: Improved $\mathsf{NC}^2$ upper bound (by Almagor, Cohen, Pérez, Shirmohammadi, and Worrell in 2020) to $\mathsf{AC}^1$

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Open: Multiple counters

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Open: Multiple counters, e.g. $S = \{(x, y) \in \mathbb{N}^2 \mid x \leqslant y \leqslant 2x\}$