STABILIZER LIMITS & ORBIT CLOSURES
in
GEOMETRIC COMPLEXITY THEORY

CAALM - 2025 - UNIVERSITÉ SORBONNE

Joint: Bharat Adsul,
Milind Sohoni

ARXIV: 2309.15816,
2201.00135

OUTLINE

1. Valiant
   - Universality of determinant

2. The GCT viewpoint
   - 1 parameter subgroup
   - Tangent of approach

3. Tools & Results
   - Lie algebras
   - Stabilizer limits
   - $\hat{\mathcal{K}} \subseteq \mathcal{H}$
   - Nilpotency / Alignment

4. BRUHAT-TITS theory
   - BOUNDED SUBGROUPS

1 VALIANT.

$$\det \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{13}x_{22}x_{31}$$

$$\det \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{aligned} & x_{11} x_{22} x_{33} \; (1)(2)(3) \\ & - x_{11} x_{23} x_{32} \; (1)(23) \\ & - x_{12} x_{21} x_{33} \; (12)(3) \\ & + x_{12} x_{23} x_{31} \; (123) \end{aligned}$$

$$+ \underset{(132)}{x_{13} x_{21} x_{32}} - \underset{(13)(2)}{x_{13} x_{22} x_{31}}$$

THEOREM (Val, 78): Every polynomial in m variables is the determinant of an $n \times n$ matrix with entries of the form $a_0 + a_1 x_1 + \cdots + a_m x_m$.

$$p(x_1, x_2, x_3, x_4, x_5, x_6) =$$

$$-8x_1 - 8x_4 + 2x_3 - x_1x_2 + 2x_3x_4 + x_1x_2x_6 - x_1x_4x_6$$
$$+ x_2x_4x_6$$

$$p(x_1, x_2, x_3, x_4, x_5, x_6) =$$

$$-8x_1 - 8x_4 + 2x_3 - x_1x_6 + 2x_3x_4 + x_1x_2x_6 - x_1x_4x_6 + x_2x_4x_6$$

$$= \det \begin{bmatrix} x_1 + x_4 & x_1 & x_3 \\ 1 + x_4 & x_2 & 4 \\ 0 & 2 & x_6 \end{bmatrix}$$

$dc(f)$ = size of the smallest determinant representing f.

$$\mathrm{perm}_m = \mathrm{perm}\begin{pmatrix} x_{11} & \cdots & x_{1m} \\ & & \\ x_{m1} & \cdots & x_{mm} \end{pmatrix}$$

$$= \sum_{\pi \in S_m} \prod x_{i\pi(i)}$$

Conj: $dc(\mathrm{perm}_m) = \omega(m^c)$, – Superpolynomial in m

GCT VIEWPOINT

- Group action on polynomials
- $P_{n,d}$: Homogeneous polynomials of deg d in n variables
- Action of $GL(n, \mathbb{C})$

<u>Ex:</u> $p = x_1^3 + x_1 x_2 x_3 \qquad g = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$

$$g \cdot p = (x_1 + 2x_2 + 3x_3)^3 + (x_1 + 2x_2 + 3x_3)(x_2)(x_1 + x_3)$$

Mulmuley & Sohoni:

If p is expressible as the determinant of an $n \times n$ matrix (as in Valiant) then p is in the orbit closure of $\det_n$

Orbits, orbit closure, geometric invariant theory, representation theory ......

GCT Viewpoint

$$\det\begin{pmatrix} x_1+x_4 & x_1 & x_3 \\ 1+x_4 & x_2 & 4 \\ 0 & 2 & x_6 \end{pmatrix}$$

$$\Big\downarrow \text{ homogenize}$$

$$Z := \det\begin{pmatrix} x_1+x_4 & x_1 & x_3 \\ x_9+x_4 & x_2 & 4x_9 \\ 0 & 2x_9 & x_6 \end{pmatrix} = \begin{array}{l} -8x_1x_9^2 - 8x_4x_9^2 \\ +2x_3x_9^2 - x_1x_6x_9 \\ +2x_3x_4x_9 + x_1x_2x_6 \\ -x_1x_4x_6 + x_2x_4x_6 \end{array}$$

Sketch of proof:

- Identify linearly independent entries.

$$\begin{pmatrix} \underline{x_1 + x_4} & \underline{x_1} & \underline{x_3} \\ x_9 + x_4 & \underline{x_2} & \underline{4x_9} \\ 0 & 2x_9 & \underline{x_6} \end{pmatrix}$$

- Find a basis of the complement

x_5, x_7, x_8

- Replace dependent terms

$$y := \det \begin{pmatrix} \underline{x_1 + x_4} & \underline{x_1} & \underline{x_3} \\ x_9 + x_4 + x_5 & \underline{x_2} & \underline{4x_9} \\ 0 + x_7 & 2x_9 + x_8 & \underline{x_6} \end{pmatrix}$$

let $g = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 4 & 0 & 2 & 0 \end{bmatrix}$ (invertible)

$g \cdot \det(x_1, \dots, x_9) = y.$

$$q(x,t) = \det \begin{pmatrix} x_1+x_4 & x_1 & x_3 \\ x_9+x_4+tx_5 & x_2 & 4x_9 \\ tx_7 & 2x_9+tx_8 & x_6 \end{pmatrix}$$

$$= \det \begin{pmatrix} x_1+x_4 & x_1 & x_3 \\ x_9+x_4 & x_2 & 4x_9 \\ tx_7 & 2x_9+tx_8 & x_6 \end{pmatrix} + \det \begin{pmatrix} x_1+x_4 & x_1 & x_3 \\ tx_5 & 0 & 0 \\ tx_7 & 2x_9+tx_8 & x_6 \end{pmatrix}$$

$$= \det \begin{pmatrix} x_1+x_4 & x_1 & x_3 \\ x_9+x_4 & x_2 & 4x_9 \\ 0 & 2x_9 & x_6 \end{pmatrix} + \det \begin{pmatrix} x_1+x_4 & x_1 & x_3 \\ x_9+x_4 & x_2 & 4x_9 \\ tx_7 & tx_8 & 0 \end{pmatrix} + \begin{pmatrix} \quad \end{pmatrix}$$

OBSERVE:

$$q(x,t) = z + t^2(\quad) + t(\quad)$$

As $t \to 0$, $q(x,t) \longrightarrow z$

$$\lambda(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\lambda(t) \cdot y = q(x,t)$$

$$\boxed{\therefore\ \lambda(t)\left[g \cdot \det\right] \xrightarrow{t=0} z}$$

RECASTING VALIANT.

For every homogeneous poly q of deg d in m variables there is an $n \geq d$, s.t $x_{nn}^{n-d} q$ is the leading term of the action of a 1-PS on a point in the orbit of $\det_n$ (ie) $\overline{O(\det_n)} \ni q$

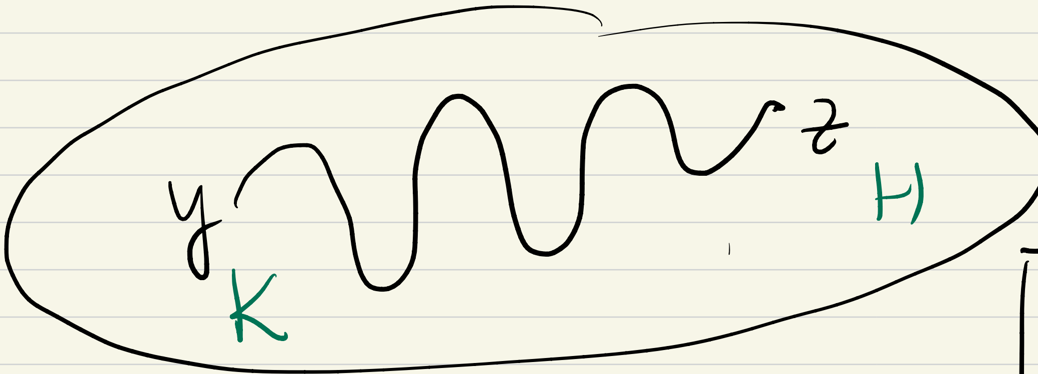
GCT approach:

1) Understand $\overline{\det_n}$ under $GL_{n^2} \circlearrowright \det_n$

2) $\mathcal{O}(\det_n) \cong GL_{n^2}/K$ → Stab of $\det_n$ $= \{ g \mid g \cdot \det_n = \det_n \}$

3) $\mathcal{O}(x_{nn}^{n-m} \operatorname{perm}_m) \cong GL_{n^2}/H$

LOWER BOUND SHOULD COME from STABILIZER DATA



$$\boxed{\hat{y} = z}$$

$$\lambda(t)\cdot y = z + t^e y_e + \cdots$$

limit

tangent of approach

Q: Understand how K & H are related

$$\text{Lie Groups} \underset{\text{functor}}{\longleftrightarrow} \text{Lie Algebras}$$

1. $\mathrm{Lie}(GL_n) = M_n$

$$[\ ]: M_n \times M_n \longrightarrow M_n$$
$$(A, B) \longmapsto AB - BA$$

2) $\mathrm{Lie}(SL_n) = \{ M \in M_n \mid \mathrm{Tr}(M) = 0 \}$

$[\ \ ] : (A, B) \longmapsto AB - BA$

3) $\mathrm{Lie}(\underset{\sim}{\text{Unipotent}}) \approx$ Nilpotent upper $\triangle$ matrices

Stabilizer Limits

- $\mathrm{Lie}(GL_{n^2}) = M_{n^2}(\mathbb{C})$
- $\lambda(t) \curvearrowright M_{n^2}(\mathbb{C})$ — Conjugation

$$\lambda(t) = \begin{pmatrix} 1 & & \\ & t^2 & \\ & & t \end{pmatrix} \quad M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\lambda(t)\cdot M = \begin{bmatrix} a_{11} & t^{-2}a_{12} & t^{-1}a_{13} \\ t^{2}a_{21} & a_{22} & t\,a_{23} \\ t\,a_{31} & t^{-1}a_{32} & a_{33} \end{bmatrix}$$

$$= t^{-2}\begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + t^{-1}\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{bmatrix} + t\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix}$$

$$+ \quad t^{2}\begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Def: For $M \neq 0$ in M_n, $\hat{M} \neq 0$ is the coeff of the least exponent of t in $\lambda(t) \cdot M$.

So if $\lambda(t) \cdot M = \sum_{a, M_a \neq 0} t^a M_a$, $\hat{M} = \min_a M_a$.

Here 0 is the all zeros matrix

Lemma: [Ass] Assume $Z = \hat{y}$ under $\lambda(H)$; $\mathcal{K} = \mathrm{Lie}(K)$; $\mathcal{H} = \mathrm{Lie}(H)$; Define $\hat{\mathcal{K}} := \{\hat{k} \mid k \in \mathcal{K}\}$. Then

1) $\dim(\hat{\mathcal{K}}) = \dim(\mathcal{K})$

2) $\hat{\mathcal{K}} \subseteq \mathcal{H}$;

3) $\hat{\mathcal{K}}$ is a Lie subalgebra of $\mathfrak{g} := M_{n^2}$

THM [ASS] $z := \hat{y}$ under $\lambda(t)$

Either:

① $\exists$ a unipotent u, $u \cdot y \xrightarrow{\lambda} \hat{z}$

and a diagonalizable $k \in \mathcal{K}$, s.t

$uku^{-1} \in \mathcal{H}$ OR

② $\hat{\mathcal{K}}$ is a Nilpotent Lie algebra.

BRUHAT-TITS theory

- Bounded subgroups of K arise from convex functions f on a root system of K. K^f
- λ gives a convex function f_λ;

<u>Q</u>: How are $\hat{K}$ & K^{f_λ} related?

- Like to believe that convex functions $\{f_{\lambda_i}\}$ and $\{K^{f_{\lambda_i}}\}$ will give information on how $\mathcal{H}(H)$ is generated from $K^{f_{\lambda_i}}$

Thank You