Structural Results for Arithmetic Formulas

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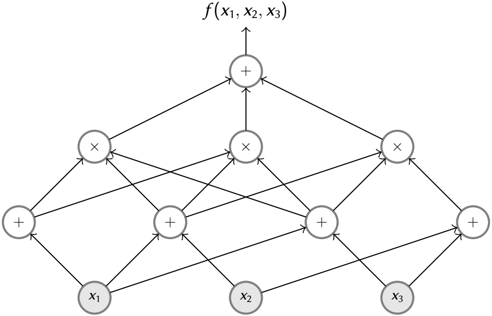
Arithmetic Circuits

Let $P(x_1, \ldots, x_N) \in \mathbb{F}[x_1, \ldots, x_N]$ be a polynomial
(In this talk $\mathbb{F}$ field of characteristic 0)

An arithmetic circuit is a model of computation which computes polynomials.



Hard to obtain lower bounds

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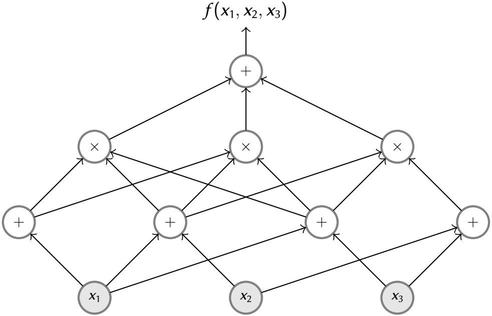
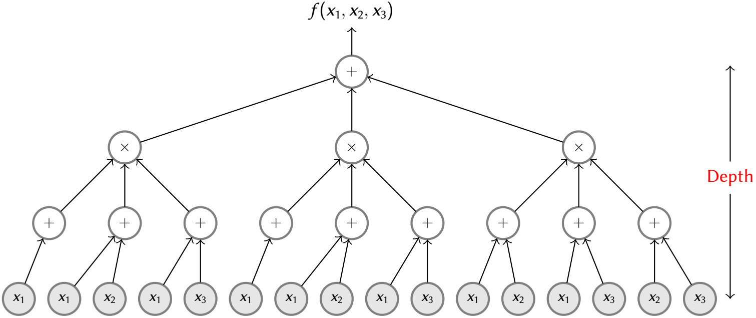


Hard to obtain lower bounds
First: lower bounds for formulas?

Arithmetic formula (graph is a tree)

Let $P(x_1, \ldots, x_N) \in \mathbb{F}[x_1, \ldots, x_N]$ be a polynomial

We can expand arithmetic circuits to get arithmetic formulas (possible blow-up: $(\text{size})^{\text{depth}}$).

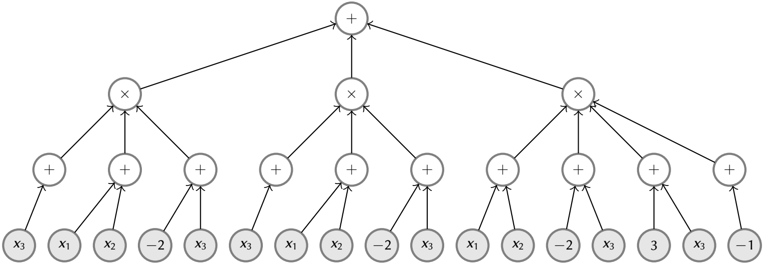


$x_1(x_1+x_2)(x_1+x_3)+x_1(x_1+x_2)(x_1+x_3)+(x_1+x_2)(x_1+x_3)(x_2+x_3)$

Size = Number of leaves. In this case 16

A formula is homogeneous if all intermediate gates are

Syntactic degree



- Degree not clear from the formula (maybe some cancelations)
- Syntactic degree:
  1. $c \in \mathbb{F}$ has degree 0
  2. x_i has degree 1
  3. Degree of a $+$ gate: max of the degrees of the children
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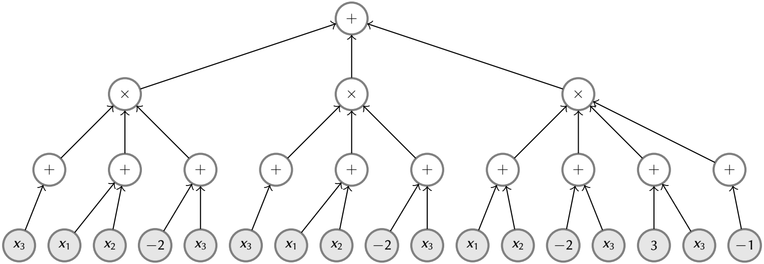

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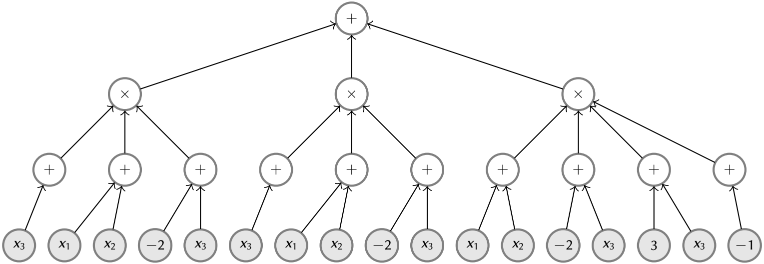

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Algebraic Complexity Theory

Algebraic lower bounds.

> **Question**
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> Prove superpolynomial lower bounds for general arithmetic circuits.

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- Superpolynomial lower bounds for constant depth arithmetic formulas for the determinant. [LST 2021]

Arithmetic formulas are surprisingly powerful

- Interpolation of coefficient of a polynomial of small degree d

$$[f(x)]_{x^p} = \sum_{i=0}^{d} \lambda_{pi} f(i)$$

- Elementary symmetric polynomials [Ben-Or]

$$e_d(x_1, \ldots, x_n) = \left[\prod_{i=1}^{n} (1 + tx_i)\right]_{t^d} \qquad \text{(not homogeneous!)}$$

- Polynomial division with remainder [BP 1985]
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- The determinant not expected to have poly-size formulas.

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- Superpolynomial lower bounds for constant depth arithmetic formulas for the determinant. [LST 2021]
- More precisely: Superpolynomial lower bounds against arithmetic formulas of depth $O(\log \log d)$ for $d \ll s$.

What is the best general depth-reduction for formulas when $d \ll s$?

- Parallelization of the formulas to depth $O(\log s)$. [BKM73]
- Parallelization of the circuits to depth $O(\log d)$. [VSBR83]
  $\rightarrow$ Parallel. formulas to depth $O(\log d)$ and new size $d^{O(\log s)}$

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- The question makes sense only for “unbounded fan-ins".
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- [FLMST 2023] True for homogeneous formulas
- $\rightarrow$ weaker condition: syntactic degree polynomially bounded (Quasi-homogeneous)

Cost of (quasi-)homogenization

Circuits can be homogeneized

$\rightarrow$ size of the homogeneous formula $d^{O(\log s)}$

Case where poly-size homogeneization is known:

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Consequence: if P of degree n has no quasi-homogeneous formula of size $n^{o(\log n)}$, then P also does not have any formula of size $\mathrm{poly}(n)$

Combining Quasi-homogeneization + depth reduction

[VSBR 1983] Reduction to $O(\log d)$: cost $d^{O(\log s)}$

[FLMST 23]
Syntactic degree is $\text{poly}(d) \to$ parallelization depth $O(\log(d))$

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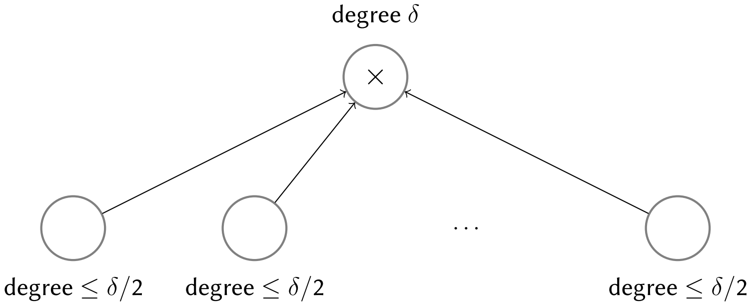
- [FLST 24] Quasi-homogeneous formula of size $d^{o(\log s)}$

  General parallel. to $O(\log d)$ but new size $d^{o(\log s)}$

Parallelization of homogeneous formulas - Intuition I

If the (syntactic) degrees of all the gates are well-distributed:



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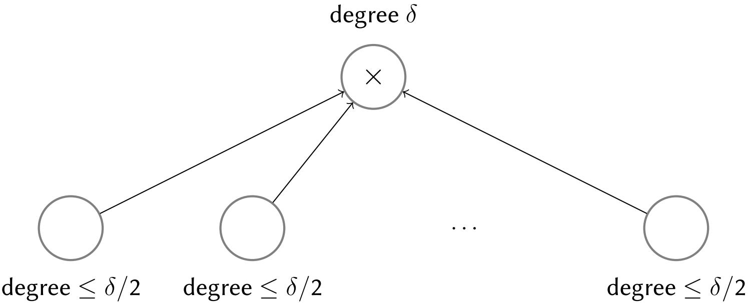

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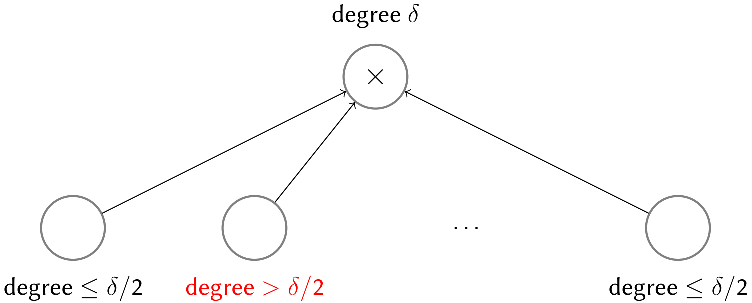

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Hard case: Below a $\times$-gate, there is one subformula of large degree.

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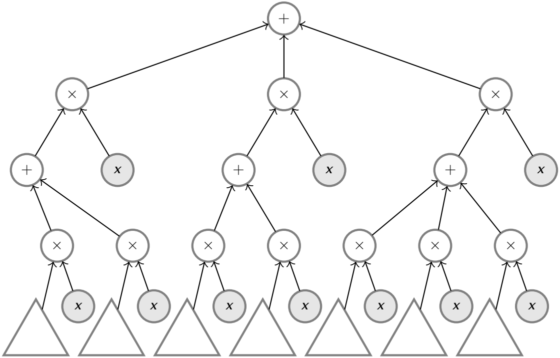

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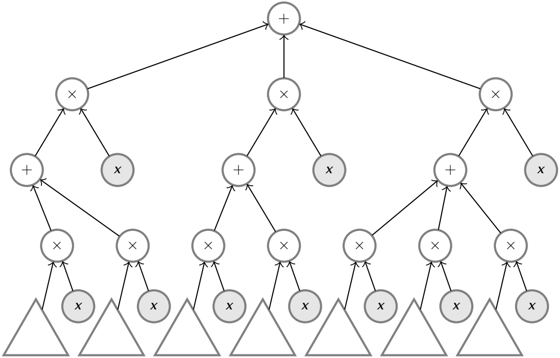

The sparsity is bounded by the number of leaves!
Small $\sum \prod$ formula!

Parallelization of homogeneous formulas - Proof

Let us associate to each node α its κ-potential:

$$\phi_\kappa(\alpha) = \lceil \log d_\alpha \rceil + \lceil \mathrm{depth}(F_\alpha)/\kappa \rceil.$$

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Lemma

Let κ be a positive integer.
Any homogeneous formula F of fan-in 2, depth Δ, and size s can be parallelized into a formula G (of arbitrary fan-in) of product-depth at most $\phi_\kappa(root)$ and size at most $s2^{\kappa \log d}$.

Starting with [BKM73] parallelization.
Then, taking $\kappa = \lceil \log s / \log d \rceil$ gives the announced parallelization.

Proof of the lemma

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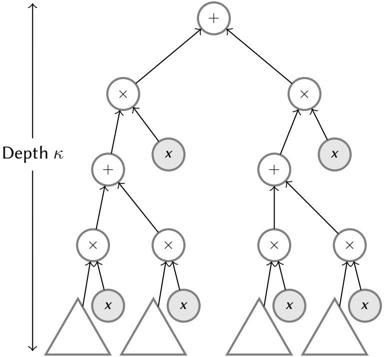

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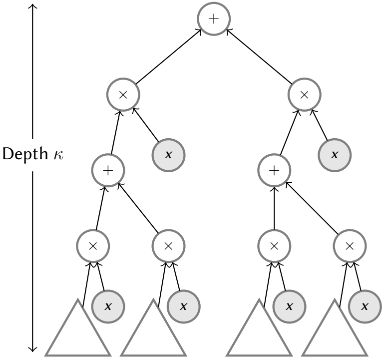

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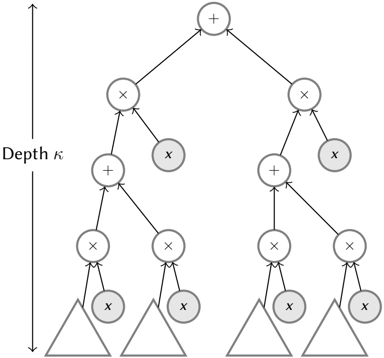

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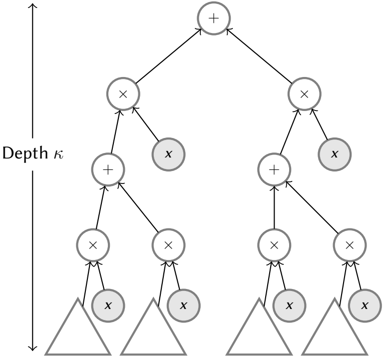

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- Leaves of the whole formula duplicated at most $2^{\kappa \log d}$ times.

Summary

Any formula of size s and syntactic degree $\text{poly}(d)$
can be parallelized into a formula with
product-depth $\leq O(\log d)$ and size $\leq \text{poly}(s)$.

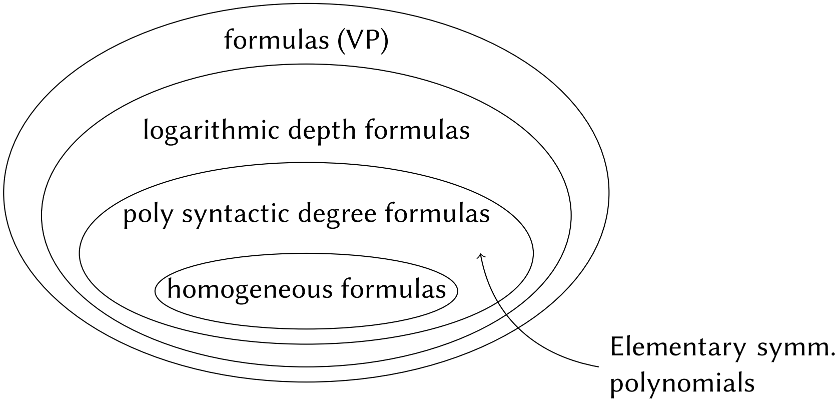
Corollary: Any formula of size s

(1) with $d = o(\log s)$, can be parallel.
to depth $\leq O(\log d)$ and size $\leq \text{poly}(s)$,

(2) can be parallel. to depth $\leq O(\log d)$ and size $\leq d^{o(\log s)}$

- It preserves monotonicity/order/(set)-multilinearity
- Optimal in the monotone case!
- Also obtain a near-linear parallelization [BB94,BCE95]:
size $s^{1+\varepsilon}$ and depth $2^{O(1/\varepsilon)} \log d$.

Open Questions



Thank you