Fair Division Beyond Monotone Valuations

Siddharth Barman

Joint work with Paritosh Verma: arXiv 2501.14609

Equitably dense subgraphs

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For any graph $G = (V, E)$ and integer $k \leq |V|$, there always exists a partition $V_1, \ldots, V_k$ such that for all $i, j \in [k]$

$$|\rho(V_i) - \rho(V_j)| \leq 4.$$

$\rho(S) :=$ edge density of subgraph induced by $S \subseteq V$

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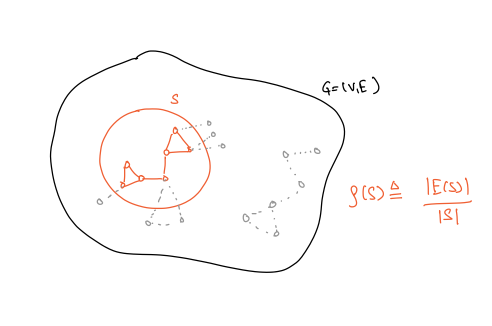

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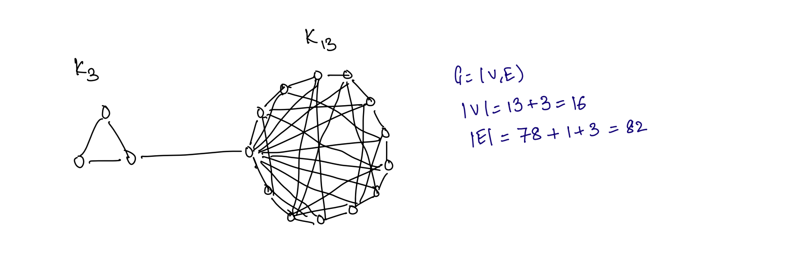

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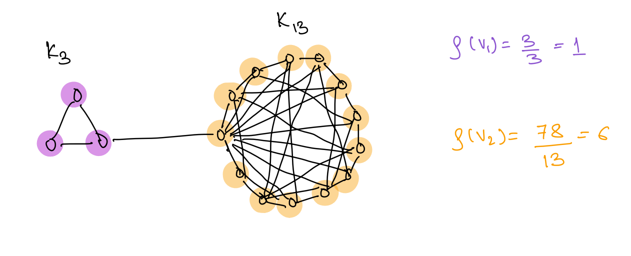

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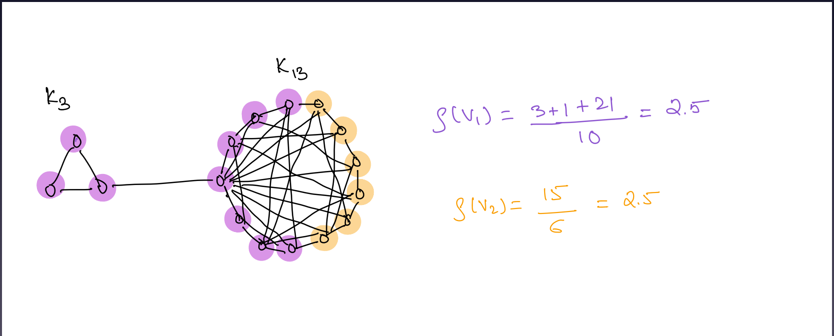

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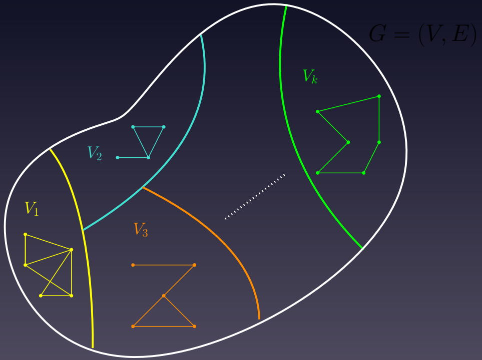

Cake Division

Cake [0,1] – divisible, heterogenous resource

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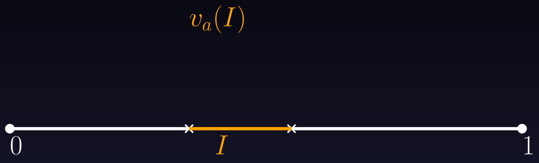
Agents' Valuations: $v_a(I)$ for agent a and interval I



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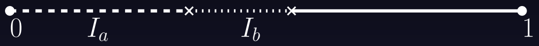

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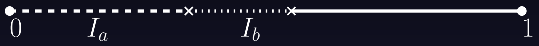

Envy-Free Division

For all agents a and b, $v_a(I_a) \geq v_a(I_b)$

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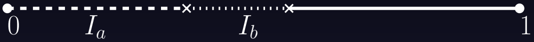

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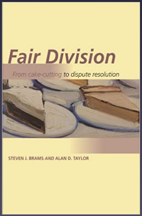

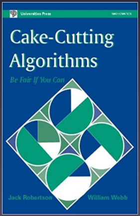

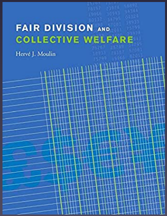

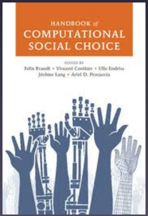

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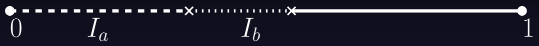

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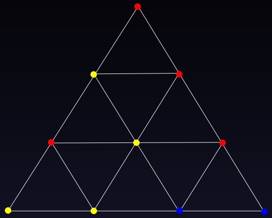
Su (1999)

An envy-free cake division always exists, under mild assumptions on the valuations.

Assumptions: v_as are continuous and bear the hungry condt.

Existence of Envy-Free Cake Divisions via Sperner's Lemma



Sperner's Lemma

Color the boundary using three colors in a legal way.
No matter how the internal nodes are colored, there exists a **trichromatic** triangle.

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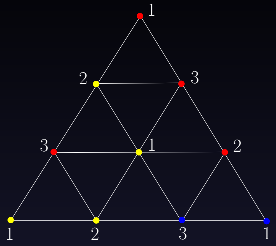

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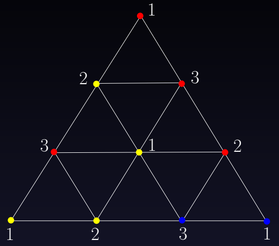

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Hungry condition: In any partition $(x_1, x_2, \ldots, x_n)$ each agent a prefers some nonempty piece

$$v_a([x_t, x_{t+1}]) > v_a(\emptyset).$$

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Aziz and Mackenzie (2016): $n^{n^{n^{n^{n^{n}}}}}$ time algorithm for envy-free cake division, with additive valuations & disconnected pieces

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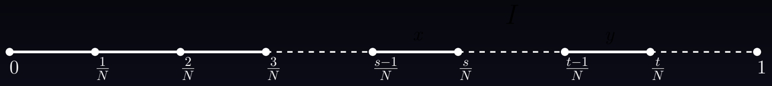

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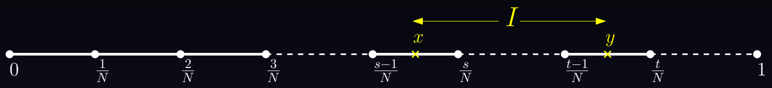

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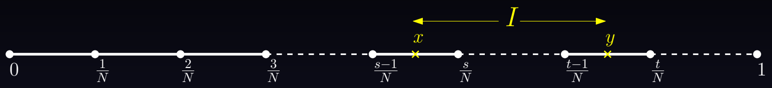

Cont. extension of the density function $\rho(\cdot)$

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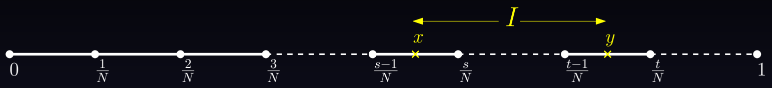

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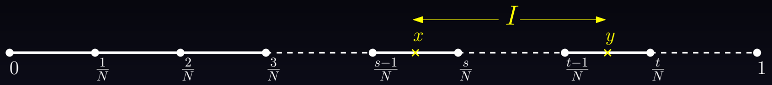

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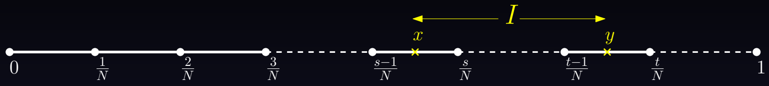

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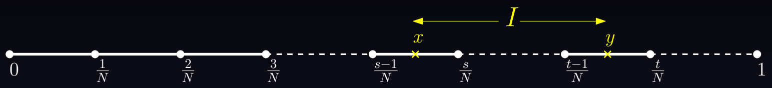

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Hungry condition ✗

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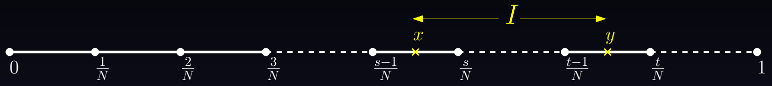

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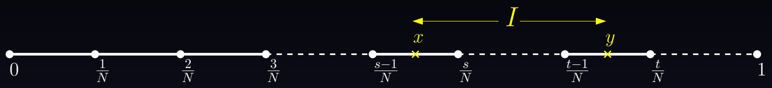

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Since v is continuous and satisfies the hungry condition, an envy-free cake division $(I_1^*, I_2^*, \ldots, I_k^*)$ always exists under v.

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Hence, from $f(I_i^*) + \varepsilon\,\mathrm{len}(I_i^*) = f(I_j^*) + \varepsilon\,\mathrm{len}(I_j^*)$ we obtain

$$f(I_i^*) \geq f(I_j^*) - \varepsilon$$

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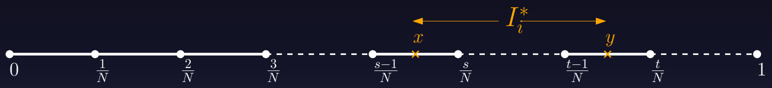

Set $V_i^* = \{s, s+1, \ldots, t-1\}$.

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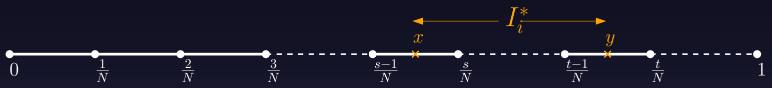

Set $V_i^* = \{s, s+1, \ldots, t-1\}$.

$$|f(I_i^*) - \rho(V_i^*)| \leq 2$$

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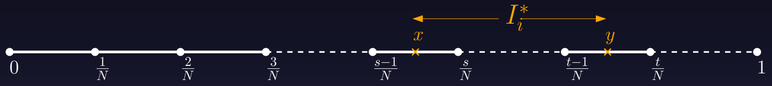

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Overall,

$$\rho(V_i^*) \geq \rho(V_j^*) - 4 - \varepsilon$$

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Polynomial-Time Algorithm ✓

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For any graph $G = (V, E)$ and $k \leq |V|$, we can efficiently find a partition $V_1, \ldots, V_k$ such that for all $i, j \in [k]$

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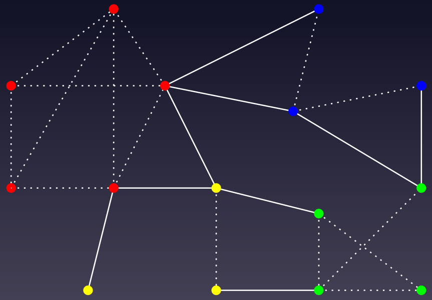
Algorithm: Within $\mathcal{F}_\varepsilon(\tau)$, find k independent intervals with maximum total length - Dynamic Program. Round.

Equitable Graph Cuts

For any graph $G = (V, E)$ and $k \leq |V|$, there always exists a partition $V_1, \ldots, V_k \neq \emptyset$ such that for all $i, j \in [k]$

$$|\delta(V_i) - \delta(V_j)| \leq 5\Delta + 1$$

$\delta()$ – cut function & Δ – max degree



$k = 4$

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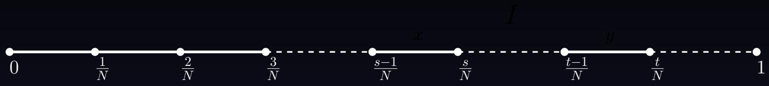

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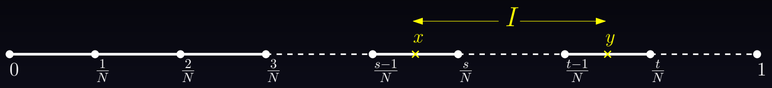

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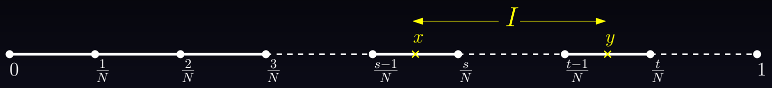

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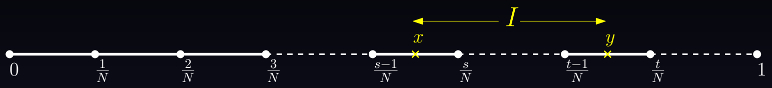

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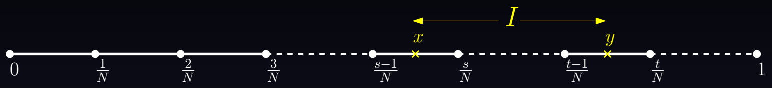

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$\Pr\{v_s \in R\} = N\left(x - \frac{s}{N}\right)$ and

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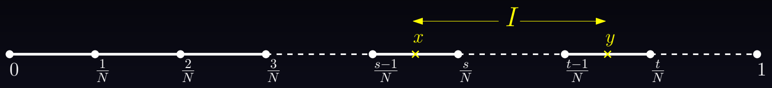
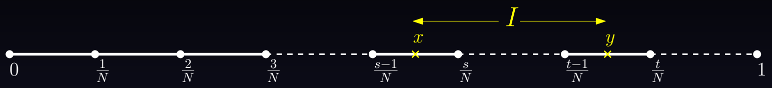
Graph $G = (V, E)$ and $k \leq N = |V|$.



Cont. extension of the cut function: $f(I)$

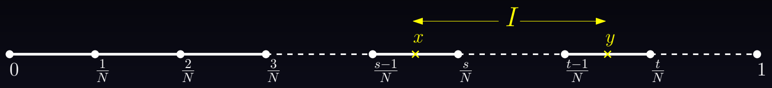
Graph $G = (V, E)$ and $k \leq N = |V|$.



Cont. extension of the cut function: $f(I)$

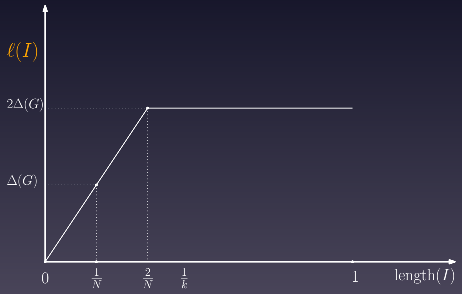
Hungry condition ✗

Graph $G = (V, E)$ and $k \leq N = |V|$.



Cont. extension of the cut function: $f(I)$

Identical valuation of k agents $v(I) = f(I) + \ell(I)$



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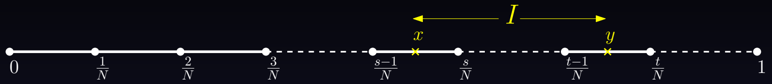

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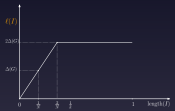

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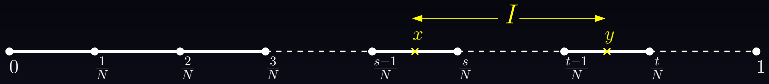

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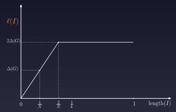
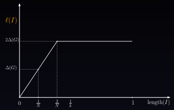


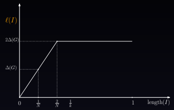
Since v is continuous and satisfies the hungry condition, an envy-free cake division $(x_1^*, x_2^*, \ldots, x_k^*)$ always exists under v.

Cont. extension of the cut fn., $f(I)$, and “hungry” fn., $\ell(I)$



Valuation $v(I) = f(I) + \ell(I)$

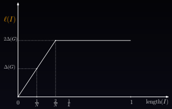
Cont. extension of the cut fn., $f(I)$, and "hungry" fn., $\ell(I)$



Valuation $v(I) = f(I) + \ell(I)$

Envy-free div $(x_1^*, \ldots, x_k^*)$, under (identical) valuation v of k agents.

Cont. extension of the cut fn., $f(I)$, and "hungry" fn., $\ell(I)$



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Envy-free div $(x_1^*, \ldots, x_k^*)$, under (identical) valuation v of k agents.

Induced intervals $I_1^*, \ldots, I_k^*$ satisfy $v(I_i^*) = v(I_j^*)$.

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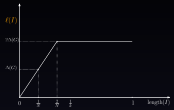

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Induced intervals $I_1^*, \ldots, I_k^*$ satisfy $v(I_i^*) = v(I_j^*)$.

Also, $\mathrm{len}(I_i^*) \geq 1/N$ for all i.

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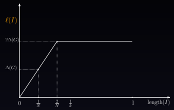

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Induced intervals $I_1^*, \ldots, I_k^*$ satisfy $v(I_i^*) = v(I_j^*)$.

Also, $\mathrm{len}(I_i^*) \geq 1/N$ for all i.

Hence, from $f(I_i^*) + \ell(I_i^*) = f(I_j^*) + \ell(I_j^*)$ we obtain

$$f(I_i^*) \geq f(I_j^*) - \Delta - 1.$$

Cont. extension of the cut fn., $f(I)$.
Intervals $I_1^*, \ldots, I_k^*$ satisfy

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Rounding: From intervals $I_1^*, \ldots, I_k^*$ to partition $V_1^*, \ldots, V_k^*$

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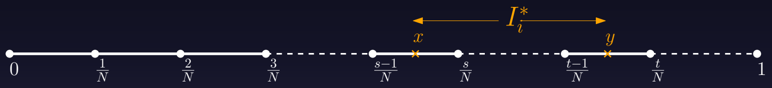

Set $V_i^* = \{s, s+1, \ldots, t-1\}$.

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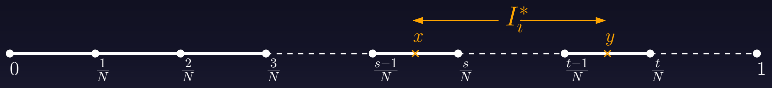

Set $V_i^* = \{s, s+1, \ldots, t-1\}$.
Note $V_i^* \neq \emptyset$.

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Intervals $I_1^*, \ldots, I_k^*$ satisfy

$$f(I_i^*) \geq f(I_j^*) - \Delta - 1.$$

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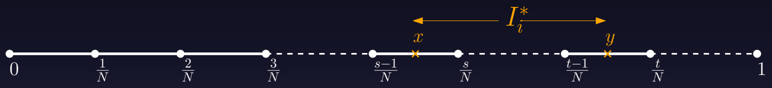

Set $V_i^* = \{s, s+1, \ldots, t-1\}$.

$$|f(I_i^*) - \delta(V_i^*)| \leq 2\Delta.$$

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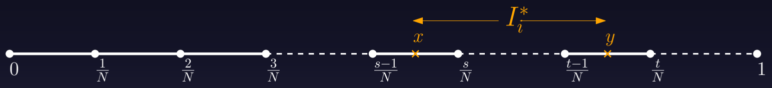

Set $V_i^* = \{s, s+1, \ldots, t-1\}$.

$$|f(I_i^*) - \delta(V_i^*)| \leq 2\Delta.$$

Overall,

$$\delta(V_i^*) \geq \delta(V_j^*) - 5\Delta - 1.$$

For any graph $G = (V, E)$ and $k \leq |V|$, there always exists a partition $V_1, \ldots, V_k \neq \emptyset$ such that for all $i, j \in [k]$

$$|\delta(V_i) - \delta(V_j)| \leq 5\Delta + 1$$

$\delta()$ – cut function & Δ – max degree

Subadditive valuation $v : 2^E \mapsto \mathbb{R}_+$
Additive cost c
Quasilinear utility $u(S) := v(S) - c(S)$.

For any quasilinear u (with $u(E) \geq 0$) and any $k \leq |E|$, there exists k-partition $E_1, \ldots E_k$ such that for all $i, j \in [k]$

$$|u(E_i) - u(E_j)| \leq 4 \operatorname{Lip}(u).$$

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Thank you!

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