Laboratoire d'Informatique Gaspard-Monge
LIGM — UMR 8049

Some recent results on (unit) multiple intervals graphs

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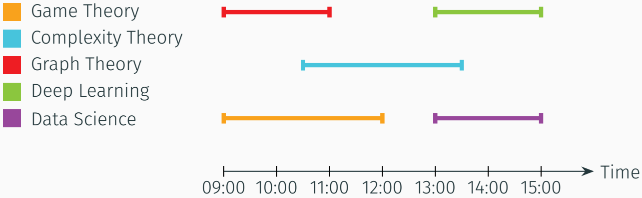
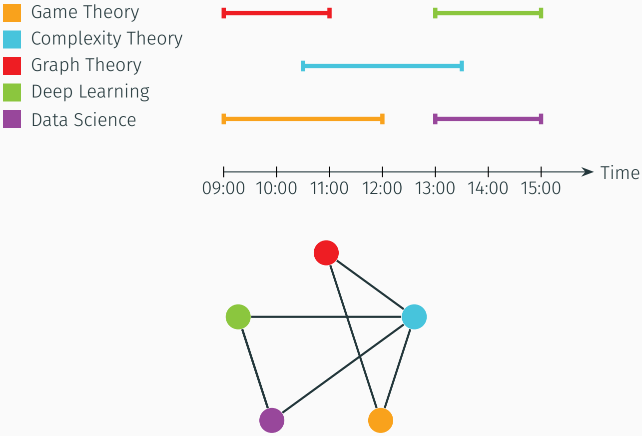
Motivating example

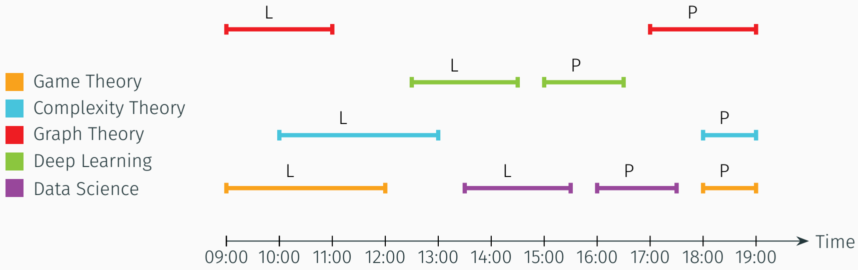
Interval graphs



Interval graphs



2-interval graphs



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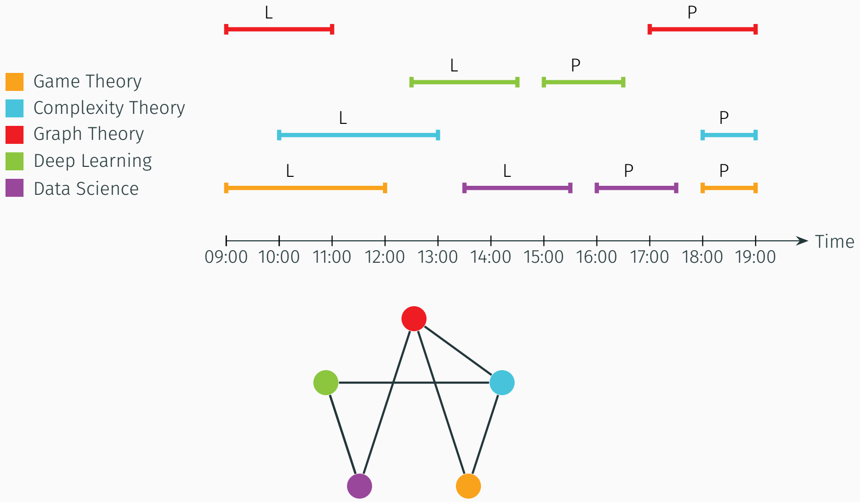
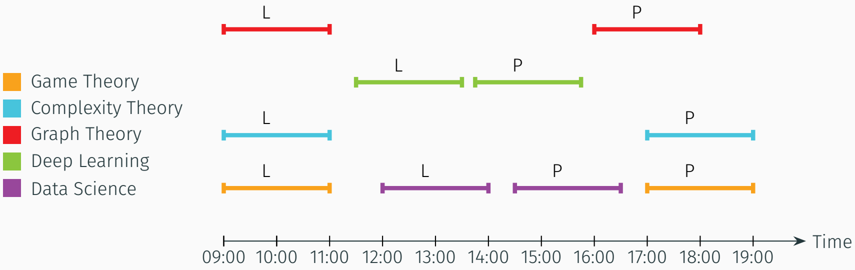
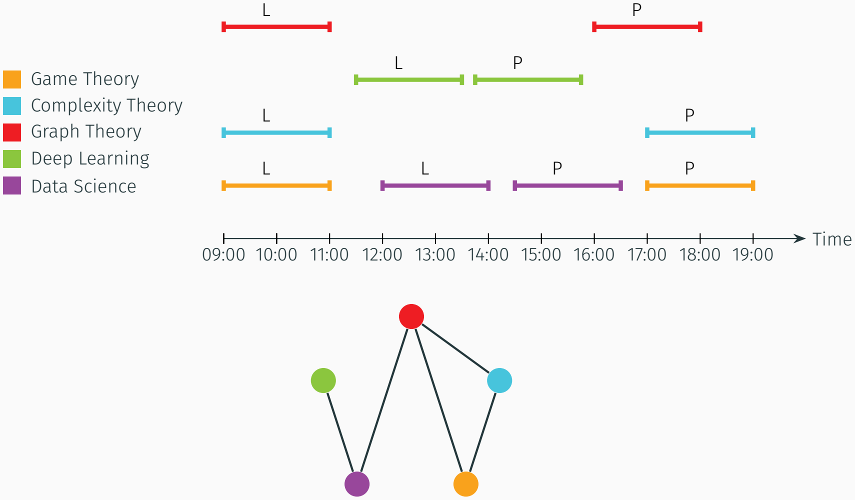


Unit 2-interval graph

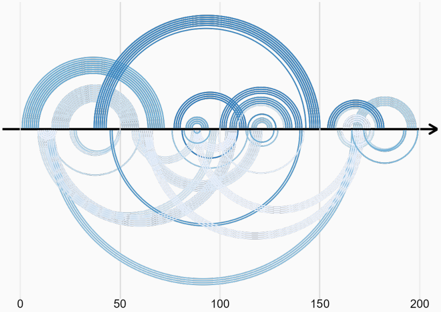


Unit 2-interval graph



Applications

- Resource allocation (interval schedulling): crew scheduling, bandwidth allocation, video-on-Demand Service, …
- High speed networks
- Storage subsystems
- Gene splitting
- RNA folding



Definitions

Interval graph

Definition

A graph is an interval graph if every vertex can be represented by an interval in such a way that there exists an edge between two vertices if and only if their corresponding intervals intersect.

Observation

Not every graph is an interval graph (eg. C_4).

Definition

An interval graph is unit if there exists an interval representation where every interval has unit length.

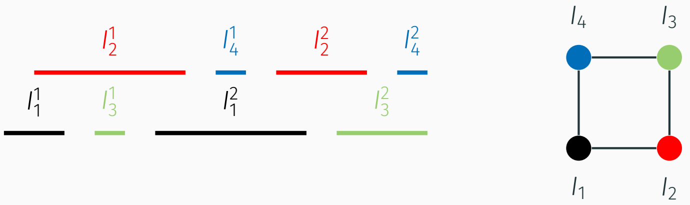
Definition

An interval graph is proper if there exists an interval representation where no interval properly contains another one.

Multiple interval graph

Definition

Let d be a positive integer. A graph G is called a (disjoint) d-interval graph if each vertex corresponds to a d-interval, and there is an edge between two vertices if and only if their respective d-intervals intersect.



Some multiple interval graph subclasses

Definition

An d-interval graph is unit if there exists an d-interval representation where every interval has unit length.

Definition

A d-interval graph is proper if there exists a d-interval representation where no interval properly contains another one.

Definition

A d-interval graph is balanced if there exists a d-interval representation where all intervals of the same d-interval have the same length.

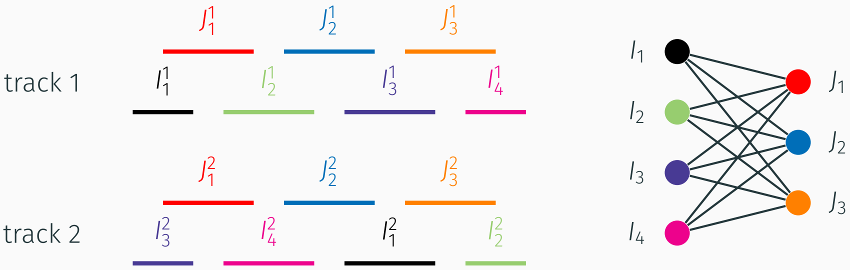
Definition

A d-interval graph is depth r if there exists a d-interval representation where at most r intervals share a common points.

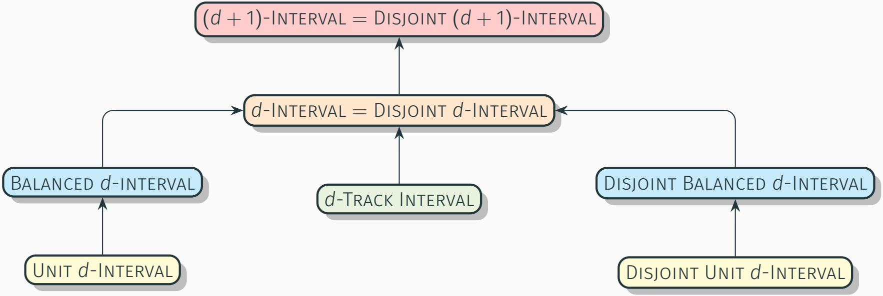
d-Track interval graph

Definition

A graph G is called a d-track interval graph if each vertex can be represented by a d-track interval — that is, a union of d intervals, one on each of d separate tracks — such that there is an edge between two vertices if and only if their corresponding d-track intervals intersect.



Graph classes



The bestiary of multiple interval graphs

- Intersection of d-dimensional boxes (boxicity, cubicity, …)
- $(\ell_1, \ell_2, \ldots, \ell_d)$-interval graphs
- $(\ell_1, \ell_2, \ldots, \ell_d)$-track interval graphs
- Mixed d-interval graphs
- Dotted interval graphs (High Throughput Genotyping)
- …

Roberts’ characterization

Roberts' characterization of unit interval graphs, '69

Theorem

For an undirected graph G, the following statements are equivalent:

1. *G is a proper interval graph.*
2. *G is a unit interval graph.*
3. *G is an interval graph that is $K_{1,3}$-free.*

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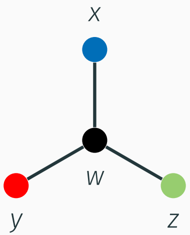

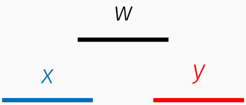

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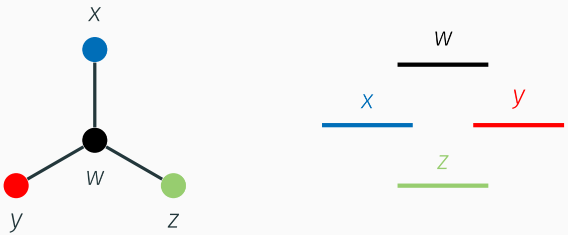


Generalizing Roberts' characterization (first attempt)

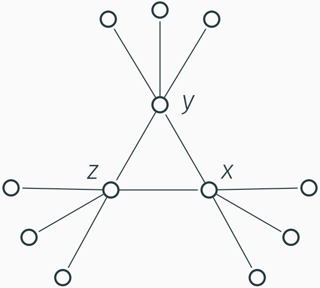
Can we characterize unit 2-interval graphs as 2-interval graphs that do not contain an induced $K_{1,5}$?

Generalizing Roberts' characterization (first attempt)

Can we characterize unit 2-interval graphs as 2-interval graphs that do not contain an induced $K_{1,5}$?

Theorem

There exists a $K_{1,5}$-free 2-interval graph that is not unit 2-interval.



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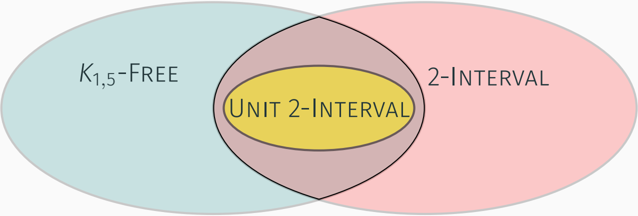

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Generalizing Roberts' characterization (second attempt)

Can we characterize unit 2-interval graphs as interval graphs that do not contain an induced $K_{1,5}$?

Yes or no … depending on the chosen definition of d-intervals.

Generalizing Roberts' characterization (second attempt)

Theorem

Let G b e an interval graph. For any $d \geq 2$, G is a unit d-interval graph if and only if G is $K_{1,2d+1}$-free.

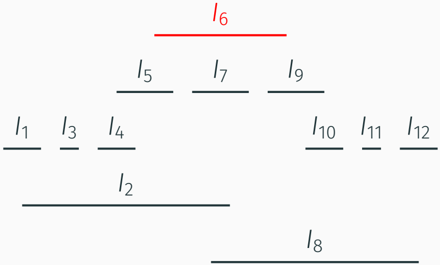
Furthermore, given a $K_{1,2d+1}$-free interval graph, a unit d-interval representation can be constructed in $O(m+n)$ time.

Theorem

Let G be a $K_{1,2d+1}$-free interval graph which does not contain any maximal $K_{1,3}$. Then, G is a **disjoint** *unit 2-interval graph.*

Generalizing Roberts' characterization (second attempt)

Interval representation of a $K_{1,5}$-free interval graph for which our algorithm does not return a disjoint unit 2-interval representation.



Disjoint unit 2-interval graphs

Theorem

There exists a $K_{1,5}$-free interval graph that is not a disjoint unit 2-interval graph.

Disjoint unit 2-interval graphs

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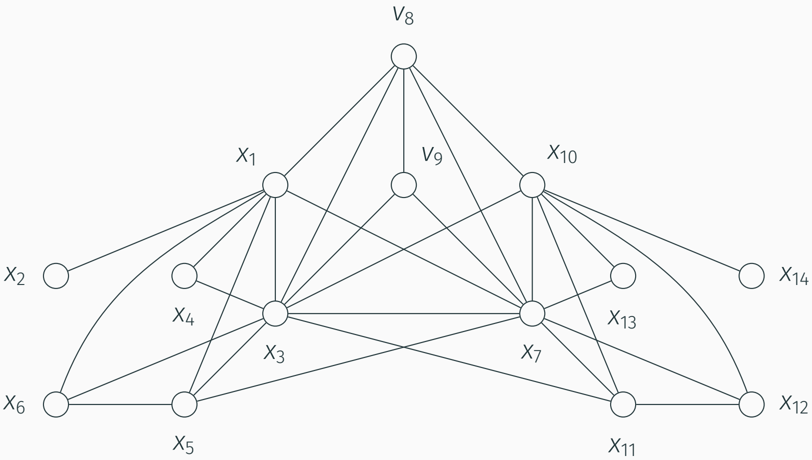

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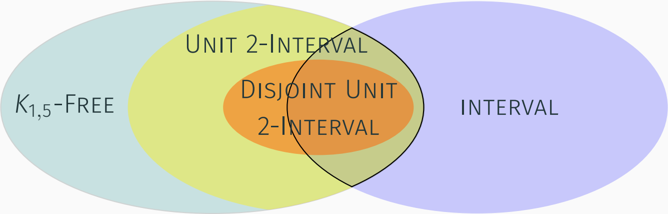


Refined graph classes

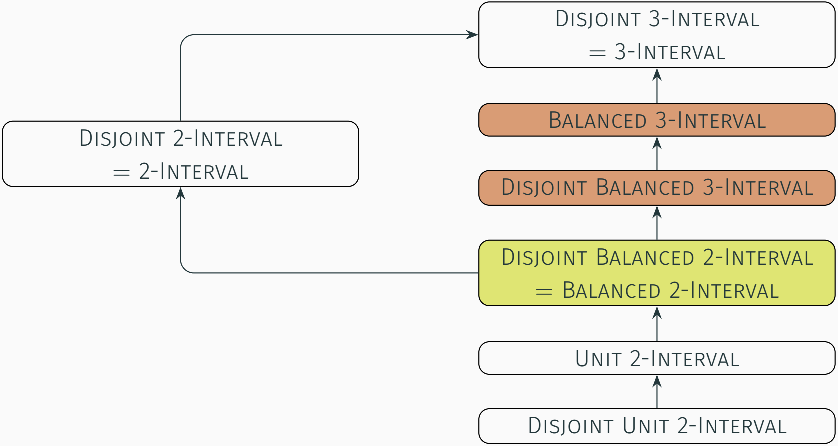
Theorem

The classes of balanced 2-interval and disjoint balanced 2-interval graphs are equivalent.

Theorem

The class of disjoint balanced 3-interval graphs is properly contained in the class of balanced 3-interval graphs.

Refined graph classes



Recognizing unit 2-interval graphs

Recognizing unit 2-interval graphs

	d-Track Interval	d-Interval
Unrestricted	NP-complete [G.W., '95; J., '13]	NP-complete [W.S., '84]
Unit	NP-complete [J., '13]	?
Depth-Two + Unit	NP-complete ($d = 2$) [J., '13]	Polynomial [J., '13]
Balanced	NP-complete [J., '13]	NP-complete ($d = 2$) [G.V., '07]

Recognizing unit 2-interval graphs

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Depth-Two + Unit	NP-complete ($d = 2$) [J., '13]	Polynomial [J., '13]
Balanced	NP-complete [J., '13]	NP-complete ($d = 2$) [G.V., '07]

Recognizing unit 2-interval graphs

Colored Disjoint Unit 2-Interval Recognition

Given a graph $G = (V, E)$ along with a vertex coloring $c : V \to \{\text{blue}, \text{black}\}$, determine whether there exists an interval representation of G satisfying the following conditions:

- each vertex colored blue is represented by a disjoint union of two unit intervals (i.e., a unit 2-interval),
- each vertex colored black is represented by a single unit interval.

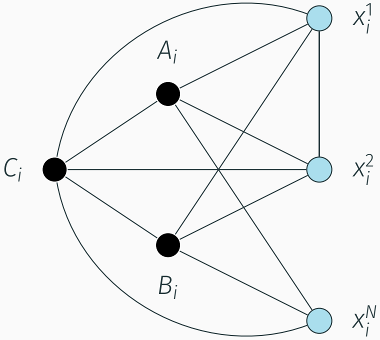
Recognizing unit 2-interval graphs

Theorem

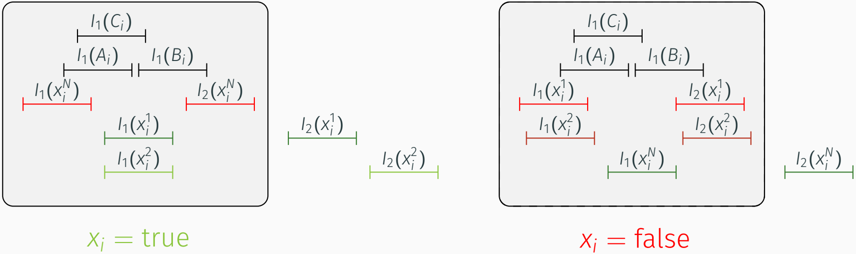
COLORED UNIT 2-INTERVAL RECOGNITION is NP*-complete, even for graphs where the white vertices have degree at most* 6 *and the black vertices have degree at most* 5*.*

- For once, we won't use CUBIC TRIANGLE-FREE HAMILTONIAN PATH.
- 3-SAT is NP-complete even if:
    - each clause has either two literals (2-clause) or three literals (3-clause),
    - each 3-clause is positive monotone,
    - each variable occurs in exactly one 3-clause, and
    - each variable occurs in exactly three clauses, once negative and twice positive.

Recognizing unit 2-interval graphs

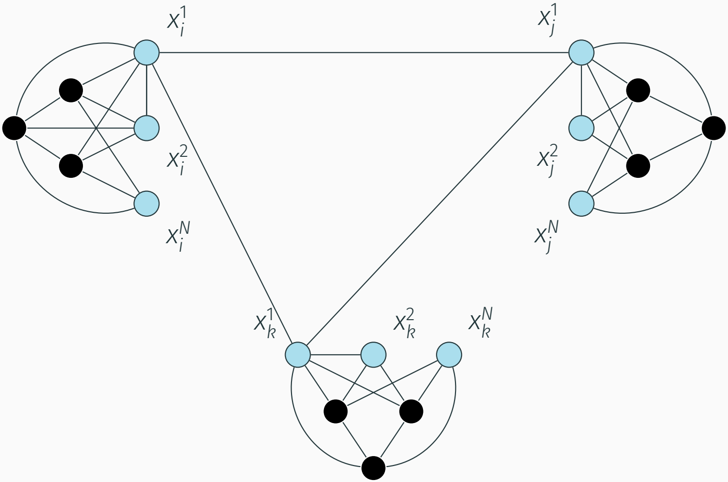


Representing variables



Representing clauses

$x_i \lor x_j \lor x_k$



Recognizing unit 2-interval graphs

Theorem

DISJOINT UNIT 2-INTERVAL RECOGNITION is NP*-complete, even for graphs with maximum degree 7.*

Theorem

DISJOINT UNIT d-INTERVAL RECOGNITION is NP*-complete for every $d \geq 2$.*

Theorem

UNIT d-INTERVAL RECOGNITION is NP*-complete for every $d \geq 2$.*

Future directions

Three open problems

- Can disjoint unit 2-interval graphs be recognized in polynomial time?
- For $d > 2$, is every unit d-interval graph also a disjoint balanced d-interval graph?
- Does there exist a constant-ratio approximation algorithm for computing the unit interval number of a graph?