K.V. Subrahmanyam, Chennai Mathematical Institute

Stabilizer Limits and Orbit Closures in Geometric Complexity Theory

Valiant showed that the determinant function is universal — any polynomial $f(X)$ can be written as $det_m(M_f)$, for a matrix M_f whose entries are affine linear functions in the variables X. M_f may be called the implementation of $f(X)$ as a determinant. A classical question has been — what is the size of the matrix M_f when f is the permanent function, $perm_m$, of an m x m matrix. It is conjectured that the matrix implementing the permanent function is of size exponential in m.

The Geometric Complexity approach casts this as an orbit closure problem. The implementation of $perm_m$ as the determinant of a matrix M_{perm_m} translates to a suitably padded permanent appearing as a point $z = perm'_m$ (with stabilizer H'_m) in the projective orbit closure of the special point $y=det_n (X)$ (with stabilizer K_n) under a 1-parameter subgroup λ acting on $Sym^n(X)$, the space of forms of degree n in the variables of X. The lower bound is expected to arise from``obstructions'', based on the stabilizers K_n and H'_m. Research has so far focussed on obstructions arising from representation theory, and this connection has yielded several negative and positive results.

We develop a Lie algebraic approach to the problem which allows a more geometric analysis. Here $\cal K$ and $\cal H$, the Lie algebras of K_n and H'_m are the central objects. Using λ-weights and leading terms, we develop a theory of how stabilizers change under taking limits. We show that the leading terms $\widehat{\cal K}$ of the Lie algebra $\cal K$, form a special Lie subalgebra of $\cal H$. This is the first known connection between the two stabilizers. We next define alignment, i.e., semisimple elements $s\in H\cap K$ and show how their presence leads to combinatorial conditions on the implementation A. We also show how attributes of $\widehat{\cal K} \rightarrow \cal H$ lead us to structures which further connect $\cal H$ and $\cal K$ and point to orbits intermediate to $O(y)$ and $O(z)$.

A central question in this approach is on the nature of alignments, conditions for the same, and their absence. An exponential lower bound for this problem by Landsberg and Ressayre when the implementation is constrained to be equivariant can be viewed as the extreme case — of complete alignment — giving hope that a nuanced understanding of alignment could lead to a proof of the lower bound.

The overall agenda is to exhibit K_n as a master group — in other words, a recipe to construct a desired stabilizer group family H'_m as a sequence of orbits and their stabilizers, starting from K_n.

Time permitting, we will show how our approach connects with many classical analyses of orbit closures, viz., of Hilbert-Mumford-Kempf, toric varieties and compactifications of Lie groups..

This is joint work with Bharat Adsul and Milind Sohoni.